A new integral and series representation of Riemann's symmetrical functional equation to prove the Riemann Hypothesis

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Abstract: Riemann's symmetrical functional equation is given by

$$\xi^*(s) := \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}\int_1^\infty \psi(x)\left[x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)}.$$

A new integral and series of $\xi^*(s)$ is given in the form

$$\xi^*(s) := \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi_\Phi(s) + \xi_\pi(s).$$

It is enabled by the paper of M. S. Milgram, (MiM), providing a new integral and series representation of the zeta function. The convergent integral representation

$$\xi_\Phi(s) := 2\int_1^\infty x^{-\frac{1}{2}}\left[\sum_{n=1}^\infty e^{-\pi n^2 x^2} - \sum_{n=1}^\infty e^{-2\pi n x}\right]\cosh\left[\left(s-\frac{1}{2}\right)\log x\right]dx$$

is like a polynomial of infinite degree, which solves the RH. The corresponding entire Zeta function in the form

$$\xi^{**}(s) := \pi^{-\frac{s}{2}}\sin\left(\frac{\pi}{2}s\right)\cos\left(\frac{\pi}{2}s\right)\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

is accompanied by a corresponding product representation in the form

$$\xi^{**}(s) = \zeta(s)\cos\left(\frac{\pi}{2}s\right)\pi^{1-\frac{s}{2}}e^{-\gamma\frac{s}{2}}\prod_{n=1}^\infty\left(1-\frac{s}{2n}\right)\left(1+\frac{s}{2n}\right)e^{\frac{s}{2n}}$$

enabling Riemann's method for deriving the formula for his prime number density function J(x), (EdH) 1.11.

A new integral and series representation of Riemann's symmetrical functional equation

Riemann's symmetrical functional equation with poles (of first order) at $s=0$ and $s=1$ is given by, (EdH) 1.7,

$$(*)\qquad \xi^*(s) := \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}\int_1^\infty \psi(x)\left[x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)} = \xi^*(1-s).$$

The integral converges for all s, because $\psi(x) = \sum_{n=1}^\infty e^{-\pi n^2 x}$ decreases more rapidly than any power of x as $x \to \infty$, (EdH) 1.7. In the critical stripe the function $\xi^*(s)$ has the same zeros as the zeta function $\zeta(s)$ (which has only a simple pole at $s=1$ with residue 1), and, as a consequence of the functional equation, it is real on the critical line.

Multiplying (*) by $s(s-1)/2$ results into the Riemann's entire Zeta function

$$\xi(s) := \Gamma\left(1+\frac{s}{2}\right)(s-1)\pi^{-\frac{s}{2}}\zeta(s) = s(s-1)\xi^*(s) = \xi(1-s),$$

from which Riemann derived his famous series representation (appendix)

$$\Xi(t) := \xi\left(\frac{1}{2}+it\right) = 4\int_1^\infty x^{-1/4}\frac{d}{dx}\left[x^{3/2}\psi'(x)\right]\cos\left(\frac{t}{2}\log x\right)dx\,.$$

Riemann's „*multiplying by* $s(s-1)/2$“ approach is in line with the „Mellin transform rules“ $M[xh'](s) = -sM[h](s)$, $M[(xg)'](s) = (1-s)M[g](s)$ resp. $M[(x^2h')'](s) = (-s)(1-s)M[h](s)$, provided that the considered integrals are convergent, (EdH) 10.3, 10.5. Corrrespondingly, the integral representation of $\Xi(t)$ is the result of a two times partial integration process accompanied by a series representation of $\Xi(t)$; without giving a proof Riemann claimed that this series is convergent, (EdH) 1.8.

We shall apply the following abbreviations

i) $\zeta^*(s) := \zeta(s)\, sin\left(\frac{\pi}{2}(1-s)\right)$

ii) $\zeta_n^*(s) := \frac{\zeta(2n)}{2n-s}$

iii) $\xi_{\pi_1}(s) := \frac{\zeta^*(s)+\zeta^*(1-s)}{sin(\pi s)} = \frac{1}{2}\left[\frac{\zeta(s)}{sin\left(\frac{\pi}{2}s\right)} + \frac{\zeta(1-s)}{cos\left(\frac{\pi}{2}s\right)}\right], \quad s \neq v\ ,\ v \in Z$

iv) $\xi_{\pi_2}(s) := \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n[\zeta_n^*(s) + \zeta_n^*(1-s)], \quad s \neq 2n, 1-2n\ ,\ n \in N$

v) $\xi_\pi(s) := \xi_{\pi_1}(s) + \xi_{\pi_2}(s)$

vi) $\varphi(x) := \frac{1}{2}\frac{e^{-\pi x}}{sinh(\pi x)} = \frac{1}{e^{2\pi x}-1} = \sum_{n=1}^{\infty} e^{-2\pi n x}$, (*)

vii) $\Phi(x) := \psi(x^2) - \varphi(x) = \sum_{n=1}^{\infty}(e^{-\pi(nx)^2} - e^{-2\pi n x})\ ,\ x \geq 1$

viii) $\xi_\Phi(s) := \int_1^\infty \Phi(x)[x^s + x^{1-s}]\frac{dx}{x} = 2\int_1^\infty \sqrt{x}\Phi(x) cosh\left[\left(s-\frac{1}{2}\right)logx\right]\frac{dx}{x}$, (**)

ix) $\xi_\Phi\left(\frac{1}{2}+iz\right) = 2\int_0^\infty \Psi(t)\cos(zt)dt$ with $\Psi(t) := e^{t/2}\sum_{n=1}^{\infty}(e^{-\pi(ne^t)^2} - e^{-2\pi n e^t})$.

Lemma:

$$-\frac{1}{2}\frac{1}{s(1-s)} = -\int_1^\infty [x^s + x^{1-s}]\varphi(x) + \xi_\pi(s)\ ,\ s \neq v\ ,\ v \in Z.$$

The proof of the lemma is given in the section below. It is based on a new integral representation of Riemann's zeta function as provided in (MiM).

From the lemma it follows

$$(**) \qquad \xi^*(s) = \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi_\Phi(s) + \xi_\pi(s)\ ,\ s \neq v\ ,\ v \in Z.$$

Multiplication by $sin(\pi s) = 2\, sin\left(\frac{\pi}{2}s\right) cos\left(\frac{\pi}{2}s\right)$ governs the singularities at $s = v$ resulting into an entire function in the form

$$\xi^{**}(s) := sin(\pi s)\, \xi^*(s) = \pi^{-\frac{s}{2}} sin\left(\frac{\pi}{2}s\right) cos\left(\frac{\pi}{2}s\right) \Gamma\left(\frac{s}{2}\right)\zeta(s)\ .$$

Applying the product representations of $sin(\pi s)$ and $\Gamma\left(1+\frac{s}{2}\right)$ (***), results into, (****)

$$\xi^{**}(s) = \zeta(s)\, cos\left(\frac{\pi}{2}s\right)\pi^{1-\frac{s}{2}}e^{-\gamma\frac{s}{2}}\prod_{n=1}^{\infty}(1-\frac{s}{2n})(1+\frac{s}{2n})\, e^{\frac{s}{2n}}\ .$$

The principle term of Riemann's density function $J(x)$ corresponds to the term $-\log(s-1)$, (EdH) 1.14. The representation in the form $(0 < Re(s) < 1)$ (*****)

$$\xi^{**}(s) = (1-s)\, sin(\pi s)\, M\left[{}_1F_1\left(\frac{1}{2};\frac{3}{2},-\pi x^2\right)\right](s)\zeta(s)$$

$$= \pi(s-1)\prod_{n=1}^{\infty}(1-\frac{s^2}{n^2})\, M\left[x\ {}_1F_1'\left(\frac{1}{2};\frac{3}{2},-\pi x^2\right)\right](s)\zeta(s)$$

provides the link to a Kummer function based zeta function theory.

(*) ; $\frac{1}{r-1} = \sum_{n=1}^{\infty} r^{-n}$; the integral $\int_1^\infty x^{2m}\varphi(x)\frac{dx}{x}$ is related to the Bernoulli numbers by the formula, (Grl) 3.552: $\int_1^\infty x^{2m}\varphi(x)\frac{dx}{x} = \frac{|B_{2m}|}{2m}$

(**) (PoG) Part V, 173: $f(t) > 0$, $f'(t) < 0$, $f''(t) < 0$ for $0 \leq t \leq 1$, then the even function $F(z) = \int_0^1 f(t)\cos(zt)dt$ has infinite many, only real zeros

(***) (LeB) p. 32: $sin\left(\frac{\pi}{2}s\right) = \frac{\pi}{2}s\prod_{n=1}^{\infty}(1-\frac{s^2}{4n^2}) = \frac{\pi}{2}s\prod_{n=-\infty, n\neq 0}^{\infty}(1-\frac{s}{2n})e^{s/2n}$, $\Gamma\left(1+\frac{s}{2}\right) = e^{-\gamma\frac{s}{2}}\prod_{n=1}^{\infty}(1+\frac{s}{2n})^{-1} e^{s/(2n)}$

(****) (ZyA) p. 5: $-\frac{1}{2\pi}\log|2\,(sin(\pi x))| = \sum_{k=1}^{\infty}\frac{\cos(2\pi k x)}{2\pi k}$, $0 < x < 1$, (Grl) 1.518: $-\log\left(cos\left(\frac{\pi}{2}x\right)\right) = -\sum_{k=1}^{\infty}\frac{(-1)^k}{2k}\frac{[2^{2k}-1]}{(2k)!}B_{2k}(\pi x)^{2k} = \sum_{k=1}^{\infty}\frac{sin^{2k}(\frac{\pi}{2}x)}{2k}$, $|x| < 1$.

(*****) (BrK): $M\left[{}_1F_1\left(\frac{1}{2};\frac{3}{2},-\pi x\right)\right]\left(\frac{s}{2}\right) = \int_0^\infty x^{\frac{s}{2}}\,{}_1F_1\left(\frac{1}{2};\frac{3}{2},-\pi x\right)\frac{dx}{x} = 2\int_0^\infty x^s\,{}_1F_1\left(\frac{1}{2};\frac{3}{2},-\pi x^2\right)\frac{dx}{x} = \pi^{-s/2}\frac{\Gamma(\frac{s}{2})}{1-s}$, $0 < Re(s) < 1$

(LeN) 9.13: $Erf(x) = x\ {}_1F_1\left(\frac{1}{2};\frac{3}{2},-x^2\right) = \int_0^x e^{-t^2}dt$.

Proof of the lemma

Lemma:

$$-\frac{1}{s(1-s)} = -\int_1^{\infty}[x^s + x^{1-s}]\frac{e^{-\pi x}}{sinh(\pi x)}\frac{dx}{x} + \left[\frac{\zeta(s)}{sin\left(\frac{\pi}{2}s\right)} + \frac{\zeta(1-s)}{cos\left(\frac{\pi}{2}s\right)}\right] + \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s}\right].$$

The proof of the lemma is based on a novel integral and series representation of the Riemann zeta function $\zeta(s)$ as provided in (MiM):

By application of the residue theorem the convergent Dirichlet series representation $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, $Re(s) > 1$, of the Riemann Zeta function can be reproduced from the contour integral representation

$$\zeta(s) = \pi\sum_{n=1}^{\infty} Res_{t=k}\oint_{t=^{\circ}k} x^{1-s}\frac{e^{i\pi x}}{sin\,(\pi x)}\frac{dx}{x}, \quad Re(s) > 1 \qquad \textit{(MiM) (2.4)}$$

Convert this representation into a contour integral representation enclosing the positive integers on the x-axis in a clockwise direction, and, provided that $Re(s) > 1$ so that contributions from infinity vanish, the contour may then be opened such that it stretches vertically in the complex x-plane, given,

$$\zeta(s) = \frac{i}{2}\int_{c-i\infty}^{c+i\infty} x^{1-s}cot(\pi x)\frac{dx}{x}, \qquad 0 < c < 1,\ c\ real \qquad \textit{(MiM) (2.6)}$$

$$= -\frac{1}{2}\int_{-\infty}^{\infty}(c+ix)^{-s}cot(\pi(c+ix))dx \qquad \textit{(MiM) (2.7)}$$

For the special case $c = 0$ the integral

$$\zeta(s) = -\pi^{s-1}\frac{sin\,(\frac{\pi}{2}s)}{s-1}\int_0^{\infty}\frac{x^{1-s}}{sinh^2(x)}dx, \qquad Re(s) < 0 \qquad \textit{(MiM) (4.1)}$$

can be broken into two parts $\zeta(s) = \zeta_0(s) + \zeta_1(s)$ where

$$\zeta_1(s) = \frac{sin\,(\frac{\pi}{2}s)}{s-1} + sin\,(\frac{\pi}{2}s)\int_1^{\infty} x^{1-s}\frac{e^{-\pi x}}{sinh(\pi x)}\frac{dx}{x} \qquad \textit{(MiM) (4.6)}$$

$$\zeta_0(s) = -\frac{2}{\pi}sin\,(\frac{\pi}{2}s)\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{2n-s} \quad (*) \qquad \textit{(MiM) (4.8)}$$

which are both valid for all s.

The lemma follows from the related identities

$$\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)} = \frac{1}{s-1} - \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{2n-s} + \int_1^{\infty} x^{1-s}\frac{e^{-\pi x}}{sinh(\pi x)}\frac{dx}{x}$$

$$\frac{\zeta(1-s)}{\sin\left(\frac{\pi}{2}(1-s)\right)} = \frac{1}{-s} - \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{(2n-1)+s} + \int_1^{\infty} x^{s}\frac{e^{-\pi x}}{sinh(\pi x)}\frac{dx}{x}.$$

(*) The reduction of the related integral $\zeta_0(s) = \frac{i}{2}\oint_{-i}^{i} x^{1-s}cot(\pi x)\frac{dx}{x}$ is enabled by the formula $xcot(\pi x) = -\frac{2}{\pi}\sum_{n=0}^{\infty} x^{2n}\zeta(2n)$, $|x| < 1$, (MiM) (4.7).

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Appendix

Some more details about Riemann's $\xi(s)$ Function

For $\psi(x) := \sum_{n=1}^{\infty} e^{-n^2 \pi x}$ the function

$$(1) \qquad \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_0^\infty x^{s/2}\psi(x)\frac{dx}{x} = \int_1^\infty \psi(x)\left[x^{s/2} + x^{(1-s)/2}\right]\frac{dx}{x} - \frac{1}{s(1-s)},$$

which occurs in the symmetrical form of the functional equation, has poles at $s = 0$ and $s = 1$ (*). The integral converges for all s because $\psi(x)$ decreases more rapidly than any power of x as $x \to \infty$, (EdH) 1.7.

(EdH) 1.8: Riemann multiplies (1) by $s(s-1)/2$, (**), and defines

$$(2) \qquad \xi(s) := \Gamma(1 + \frac{s}{2})(s-1)\pi^{-s/2}\zeta(s).$$

Then $\xi(s)$ is an entire function – that is an analytic function of s which is defined for all values of s – and the functional equation of the zeta function is equivalent to $\xi(s) = \xi(1-s)$. In combination with (1) this results into the representation in the form

$$\xi(s) = \frac{1}{2} - \frac{s(1-s)}{2}\int_0^\infty \left[x^{s/2} + x^{(1-s)/2}\right]\psi(x)\frac{dx}{x}$$

from which Riemann derived his final formula by two times partial integration in the form, (EdH) 1.8,

$$(3) \qquad \xi(s) = 4\int_1^\infty \frac{d}{dx}\left[x^{\frac{3}{2}}\psi'(x)\right]x^{-\frac{1}{4}}cosh\left[\frac{1}{2}(s - \frac{1}{2})\,log\,x\right]dx = \sum_{n=0}^\infty a_{2n}(s - \frac{1}{2})^{2n}$$

with

$$a_{2n} := 4\int_1^\infty x^{-1/4}\frac{(\frac{1}{2}log\,x)^{2n}}{(2n)!}\frac{d}{dx}\left[x^{3/2}\psi'(x)\right]dx\ .$$

Remark: In the notion of Riemann the integral form (3) is denoted in the form

$$\Xi(t) := \xi(\frac{1}{2} + it) = 4\int_1^\infty x^{-1/4}\frac{d}{dx}\left[x^{3/2}\psi'(x)\right]cos(\frac{t}{2}log\,x)dx.$$

Remark (EdH) 1.8: "*Riemann states that this series representation of as an even function of* $s - \frac{1}{2}$ *"converges very rapidly," but he gives no explicit estimates … the statements that this series "converges very rapidly," is also a statement that* $\xi(s)$ *is like a polynomial of infinite degree – a finite number of terms gives a very good approximation in any finite part of the plane. Hadamard proved that the rapid decrease of the coefficients* a_{2n} *is neccessary and sufficient to the validity of Riemann's infinite product formula of* $\xi(s)$*".*

Based on this statement (***), in combination with the approximation function $= \frac{T}{2\pi}log\frac{T}{2\pi} - \frac{T}{2\pi}$ for the number of roots of $\Xi(t) = 0$ whose real parts lie between 0 and T, Riemann conjectured that all roots are real.

(*) For $Re(s) < 0$ the zeta function is holomorph with (trivial) zeros (of first order) for $s = -2n$, $n \geq 1$. The non-trivial zeros lie in the critical stripe and are identical to the zeros of $\xi(s)$, while the functions $\Gamma(1-s)$ and $\sin(\frac{\pi}{2}s)$ have no zeros in the critical stripe. As the middle point of the zeros s_0 and $1 - s_0$ is given by $\frac{s_0 + (1-s_0)}{2} = \frac{1}{2}$ it follows $\xi\left(\frac{1}{2} + s\right) = \xi\left(\frac{1}{2} - s\right)$ and $\overline{\xi\left(\frac{1}{2} + s\right)} = \xi\left(\frac{1}{2} - \bar{s}\right)$. Because of $\sum_{n=1}^\infty r^{-n} = \frac{1}{r-1}$ for $\varphi(x) := \sum_{n=1}^\infty e^{-nx}$ and $Re(s) > 1$ one gets the representation $\Gamma(s)\zeta(s) = \int_0^\infty x^s\varphi(x)\frac{dx}{x} = \int_0^\infty \frac{x^s}{e^x - 1}\frac{dx}{x}$. Riemann's extension in the form $\int_{+\infty}^{+\infty}\frac{(-x)^s}{e^x-1}\frac{dx}{x}$ is valid for $s \in C$ and is equal to Dirichlet's function $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ for $Re(s) > 1$, (EdH) 1.4. We note Riemann's orginal contour integral representation in the form $\zeta(s) = \frac{\Gamma(1-s)}{2\pi i}\oint \frac{x^s}{e^{-x}-1}\frac{dx}{x}$, where the contour of integration encloses the negative x-axis, looping from $x = -\infty - i0$ to $x = -\infty + i0$ enclosing the point $x = 0$.

(**) We note the rules $M[xh'](s) = -sM[h](s)$, $M[(xg)'](s) = (1-s)M[g](s)$ resp. $M[(x^2h')'](s) = (-s)(1-s)M[h](s)$ provided that the considered integrals are convergent; multiplication with s resp. with $(1-s)$ „governs" a singularity of first order at $s = 0$ resp. $s = 1$.

(***) "*This function* $\Xi(t)$ *is finite for all finite* t*, and allows itself to be developed in powers of* t^2 *as very rapidly converging series*".

The connection between $\zeta(s)$ and the primes
(EdH) 1.11

The essence of the relationship between $\zeta(s)$ and the prime numbers is the Euler product formula

(1) $$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} = \prod_p \frac{p^s}{p^s-1} \quad , \qquad Re(s) > 1,$$

in which the product on the rigth is over all prime numbers p. The Möbius inverse transform is simply the inverse transform of the product formula, (EdH) 10.9,

$$\zeta(s)\prod_p(1-\frac{1}{p^s}) = 1 \text{ with } \prod_p(1-\frac{1}{p^s}) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}.$$

Taking the log of both sides of (1) and using the series $-\log(1-x) = log\frac{1}{1-x} = \sum_{n=1}^{\infty}\frac{1}{n}x^n$ puts this in the form

$$\log\zeta(s) = \sum_p\left[\sum_{n=1}^{\infty}\frac{1}{n}p^{-ns}\right] \quad , \ Re(s) > 1.$$

Since the double series on the right is absolute convergent for $Re(s) > 1$, the order of summation is unimportant and the sum can be written simply

(2) $$\log\zeta(s) = \sum_p p^{-s} + \frac{1}{2}\sum_p p^{-2s} + \frac{1}{3}\sum_p p^{-3s} + \frac{1}{4}\sum_p p^{-4s} + \cdots. \quad , \ Re(s) > 1.$$

The terms p^{-ns} are replaced by $s\int_{p^n}^{\infty}x^{-s-1}dx$ leading to

$$\frac{\log\zeta(s)}{s} = \int_0^{\infty}x^{-s-1}J(x)dx$$

It will be convenient in what follows to write this sum as a Stieltjes integral

(3) $$\log\zeta(s) = \int_0^{\infty}x^{-s}dJ(x) \quad , \ Re(s) > 1 ,$$

where $J(x)$ is the function which begins at 0 for $x = 0$ and increases by a jump of 1 at the primes p, by a jump $\frac{1}{2}$ at prime squares p^2, by a jump of $\frac{1}{3}$ at prime cubes, etc. As it is usual in the theory of Stieltjes integrals, the value of $J(x)$ at each jump is defined to be halfway between ist new value and ist old value. Thus $J(x)$ is zero for $0 \le x < 2$, is 1 for $2 < x < 3$, is $\frac{3}{2}$ for $x = 3$, is 2 for $3 < x < 4$, is $2\frac{1}{4}$ for $x = 4$, is $2\frac{1}{2}$ for $4 < x < 5$, is 3 for $x = 5$, is $3\frac{1}{2}$ for $5 < x < 7$, etc. A formula for $J(x)$ is

$$J(x) = \frac{1}{2}\left[\sum_{p^n<x}\frac{1}{n} + \sum_{p^n\le x}\frac{1}{n}\right].$$

Riemann stated the formula (3) in the slightly different form (obtained by integration of parts)

(4) $$\log\zeta(s) = s\int_0^{\infty}x^{-s}J(x)\frac{dx}{x} \quad , Re(s) > 1 \ .$$

This integral can be considered to be an ordinary Riemann integral and the formula itself can be derived without using Stieltjes integration by setting $p^{-ns} = s\int_{p^n}^{\infty}x^{-s}\frac{dx}{x}$, $Re(s) > 1$ in (1), which is Riemann's derivation of (3). Formulas (2)-(4) should be thought of as minor variations of the Euler product formula, which is the basic idea connection $\zeta(s)$ and primes.

Note: We note that the distribution function $J(x)$ is zero for $0 \le x < 2$ in line with the Chebyshev approximation function $\pi(x) \sim \int_2^x \frac{dt}{logt}$, defining the $\log\zeta(s)$ Fourier inverse function for $Re(s) > 1$ while the extended $\zeta(s)$ function to the critical stripe containing all non-trivial zeros of the zeta function, governing any appropriately defined prime number density function.