The d’Alembert paradox

For the following we refer to [GeD], [ShM]:

The root causes of this paradox are unrealistic "fluid" interaction assumptions for classical fluid equation models. “The d’Alembert “paradox” is about the failure of the Euler equation (the model of an ideal incompressible fluid) as a model for fluid-solid interaction.

"*In an ideal incompressible fluid, bodies moving at constant velocities do not experience any drag, or lift*" ([GeD]).

This is the consequence of the fact that "*incompressible potentials generate no force on obstacles*".

In other words the difficulty with ideal fluids and the source of the d’Alembert paradox is that in incompressible fluids there are no frictional forces. Two neighboring portions of an ideal fluid can move at different velocities without rubbing on each other, provided they are separated by streamline. It is clear that such a phenomenon can never occur in a real fluid, and the question is how frictional forces can be introduced into a model of a fluid.

In order to get out of the d’Alembert “paradox” one added “kinematic viscosity” (which is the inverse of the Reynolds number) as a constant to the additional “diffusion term” to the Navier-Stokes equations.

This is due to the fact that the curl-free condition is not preserved by the NSE in domains with boundaries. But in most experiments the kinematic constant is very small. Hence the Euler equation (i.e. if the kinematic constant is zero) should be a good approximation. For smooth solutions in domains without boundaries, this is true. In domains with boundaries the situation is not clear. The problem comes from the boundary conditions. In case of the NSE (viscosity constant non equal zero) it requires the classical no-slip condition. In case of the Euler equation (viscosity constant equal zero) one need to relax the no-slip condition to the condition that the normal derivative of the equation solution vanishes at the boundary.

The *Curl* and the *Helicity* operators play fundamental roles in the Euler and Navier-Stokes Equations. A properly defined extension of them to the boundary is governed resp. jeopardized by the definition of the (standard) directional derivative. Plemelj's alternative definition of an directional derivative on the boundary enables extended Green identities with less regulatory assumption to the potential, defined by the Laplacian operator.

Corresponding definitions with respect to the *Curl* and *Helicity* operators are proposed as one new conceptual solution element to answer the "classical solution problem of the NSE and the Euler equations".

The Euler equation in combination with fast decay conditions allows integrating by parts "up to infinity". This implies the d'Alembert paradox, as the curl-free condition is preserved by Euler. In case of a plane, when the plane reaches its cruise speed, the conditions of the theorem (*) above are fulfilled (up to a change of frame). The NSE does not preserve the curl-free condition, which is the good news, as it allows to get of the d'Alembert paradox, but the alternative NSE model with viscosity constant, with boundaries and with initial curl-free condition does not generate appropriate curl over time.

This leads to boundary layer theory to analyze the impact of a boundary layer on the asymptotic as the kinematic constant tends to zero.

Prandtl introduced curvilinear coordinates near the boundary to approximate the solution of the Euler equation with a kind of “boundary layer corrector”. In order to prove well-posedness (locally in time and globally under further conditions) the choice of the functional spaces is crucial. In a Sobolev framework the asymptotic does not always hold in H_1 which relies on Rayleigh instability.

Under the assumption of ideal incompressible fluids no aircraft would be able to fly:

Prandtl's mathematical model of airfoil uplift forces (with its underlying concept of "vortex lines" to model "circuit forces") is well posed as singular integral operator (PDO) equation with domain in a negative Hilbert scale.

From [ShM] we quote: “*It is not known, whether a weak solution of the full non-linear, non-stationary NSE constructed in the framework of Sobolev function spaces is unique. It has been proven that whenever a weak solution is in a space* $L(p,q)$ *the crucial measure of how near it comes to a classical solution is the (Serrin) quantity*

$$s=\frac{3}{p}+\frac{2}{q}\cdot$$

The smaller this is, the better the solution is. When $s=3/2$ *a weak solution exists. The critical value for uniqueness is* $s=1$*. A result of Serrin shows that for* $s=1$ *two existing weak solutions with the same initial values are identical. It also has been shown that if there is a weak solution satisfying a little bit more, i.e.*

$$s<1$$

then all weak solutions are actually strong, and, indeed, are infinitely differentiable with respect to both q and t“.

References

[GeD] Gérard-Varet D., Some mathematical aspects of fluid-solid interaction, Institut de Mathématiques de Jussieu, Université Paris, 7, 2012

[ShM] Shinbrot M., Lectures on Fluid Mechanics, Dover Publications Inc., Mineola, New York, 2012.