There are certain relations between the spaces $\{H_\alpha | \alpha \geq 0\}$ for different indices:

Lemma: Let $\alpha < \beta$. Then

$$\|x\|_\alpha \leq \|x\|_\beta$$

and the embedding $H_\beta \to H_\alpha$ is compact.

Lemma: Let $\alpha < \beta < \chi$. Then

$$\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu \text{ for } x \in H_\gamma$$

with $\mu = \frac{\gamma - \beta}{\gamma - \alpha}$ and $\nu = \frac{\beta - \alpha}{\gamma - \alpha}$.

Lemma: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x - y\|_\alpha \leq t^{\beta - \alpha} \|x\|_\beta$

ii) $\|x - y\|_\beta \leq \|x\|_\beta$, $\|y\|_\beta \leq \|x\|_\beta$

iii) $\|y\|_\gamma \leq t^{-(\gamma - \beta)} \|x\|_\beta$.

Corollary: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x - y\|_\rho \leq t^{\beta - \rho} \|x\|_\beta$ for $\alpha \leq \rho \leq \beta$

ii) $\|y\|_\sigma \leq t^{-(\sigma - \beta)} \|x\|_\beta$ for $\beta \leq \sigma \leq \gamma$.

Remark: Our construction of the Hilbert scale is based on the operator A with the two properties i) and ii). The domain $D(A)$ of A equipped with the norm

$$\|Ax\|^2 = \sum_{i=1} \lambda_i^2 (x, \varphi_i)^2$$

turned out to be the space H_2 which is densely and compactly embedded in $H = H_0$. It can be shown that on the contrary to any such pair of Hilbert spaces there is an operator A with the properties i) and ii) such that

$$D(A) = H_2 \quad R(A) = H_0 \text{ and } \|x\|_2 = \|Ax\|.$$

We give three examples of differential operator and singular integral operators, whereby the integral operators are related to each other by partial integration:

Example 1: Let $H = L^2(0,1)$ and

$$Au := -u''$$

with

$$D(A) = \dot{W}_2^2(0,1) := \overset{\circ}{W}{}_2^1(0,1) \cap W_2^2(0,1) \cdot$$

Building on the orthogonal set of eigenpairs $\{\lambda_i, \varphi_i\}$ of A_i , i.e.

$$-\varphi_i'' = \lambda_i \varphi_i \qquad \varphi_i(0) = \varphi_i(1) = 0$$

it holds the inclusion

$$D(A) \subseteq H_A = H_1 = \overset{\circ}{W}{}_2^1(0,1) \subseteq L^2(0,1) \cdot$$

Example 2: Let $H = L_{22}^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Then H is the space of integrable periodic function in R . Let

$$(Au)(x) := -\oint \log 2\sin\frac{x-y}{2} u(y)dy =: \oint k(x-y)u(y)dy$$

and

$$D(A) = H = L_{22}^*(\Gamma) \ \cdot$$

The Fourier coefficients of this convolution are

$$(Au)_\nu = k_\nu u_\nu = \frac{1}{2|\nu|} u_\nu$$

i.e. it holds $D(A) \subseteq H_A = H_{-1/2}(\Gamma)$.

A relation of this Fourier representation to the fractional function is given by

$$x - [x] - \frac{1}{2} = -\sum_1^\infty \frac{\sin 2\pi\nu x}{\pi\nu}$$

Remark: We give some further background and analysis of the even function

$$k(x) := -\ln\left|2\sin\frac{x}{2}\right| =: -\log\left|2\sin\frac{x}{2}\right| .$$

Consider the model problem

$$-\Delta U = 0 \qquad \text{in } \Omega$$

$$U = f \qquad \text{on } \Gamma := \partial\Omega \ ,$$

whereby the area Ω is simply connected with sufficiently smooth boundary. Let $y = y(s) - s \in (0,1]$ be a parametrization of the boundary $\partial\Omega$. Then for fixed $\bar{z}$ the functions

$$U(\bar{x}) = -\log|\bar{x} - \bar{z}|$$

Are solutions of the Lapace equation and for any $L_1(\partial\Omega)-$ integrable function $u = u(t)$ the function

$$(Au)(\bar{x}) := \oint_{\partial\Omega} \log|\bar{x} - u(t)| dt$$

is a solution of the model problem. In an appropriate Hilbert space H this defines an integral operator ,which is coercive for certain areas Ω and which fulfills the Garding inequality for general areas Ω. We give the Fourier coefficient analysis in case of $H = L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $x(s) := (\cos(s), \sin(s))$ be a parametrization of $\Gamma := S^1(R^2)$ then it holds

$$|x(s) - x(t)|^2 = \left|\begin{pmatrix}\cos(s) - \cos(t)\\ \sin(s) - \sin(t)\end{pmatrix}\right|^2 = 2 - 2\cos(s-t) = 2(1 - \cos(2\frac{s-t}{2})) = 2\left[2\sin^2\frac{s-t}{2}\right] = 4\sin^2\frac{s-t}{2}$$

and therefore

$$-\log|x(s) - x(t)| = -\log 2\left|\sin\frac{s-t}{2}\right| = k(s-t) .$$

The Fourier coefficients k_ν of the kernel $k(x)$ are calculated as follows

$$k_\nu := \frac{1}{2\pi}\oint k(x)e^{-i\nu x}dx = \frac{1}{2\pi}\int_0^{2\pi}\log\left|2\sin\frac{t}{2}\right|e^{-i\nu t}dt = \frac{2}{2\pi}\int_0^{\pi}\log\left|2\sin\frac{t}{2}\right|\cos(\nu t)dt = k_{-\nu}$$

As $\varepsilon\log 2\sin\frac{\varepsilon}{2} \underset{\varepsilon\to 0}{\longrightarrow} 0$ partial integration leads to

$$k_\nu = \frac{1}{\nu\pi}\sin(\nu t)\Big|_0^{\pi} - \frac{1}{\nu\pi}\int_0^{\pi}\frac{2\sin(\nu t)\cos\frac{t}{2}}{2\sin\frac{t}{2}}dt = -\frac{1}{\nu\pi}\int_0^{\pi}\frac{\sin(\frac{2\nu+1}{2}t) - \sin(\frac{2\nu-1}{2}t)}{2\sin\frac{t}{2}}dt$$

$$k_\nu = -\frac{1}{\nu\pi}\int_0^{\pi}(\left[\frac{1}{2} + \cos(t) .. + \cos(\nu t)\right] - \left[\frac{1}{2} + \cos(t) .. + \cos((\nu-1)t)\right])dt = -\frac{1}{\nu} .$$

Extension and generalizations

For $t > 0$ we introduce an additional inner product resp. norm by

$$(x, y)_{(t)}^2 = \sum_{i=1} e^{-\sqrt{\lambda_i} t}(x, \varphi_i)(y, \varphi_i)$$

$$\|x\|_{(t)}^2 = (x, x)_{(t)}^2 \ .$$

Now the factor have exponential decay $e^{-\sqrt{\lambda_i} t}$ instead of a polynomial decay in case of λ_i^α .
Obviously we have

$$\|x\|_{(t)} \leq c(\alpha, t)\|x\|_\alpha \text{ for } x \in H_\alpha$$

with $c(\alpha, t)$ depending only from α and $t > 0$. Thus the $(t) - norm$ is weaker than any $\alpha - norm$. On the other hand any negative norm, i.e. $\|x\|_\alpha$ with $\alpha < 0$, is bounded by the $0 - norm$ and the newly introduced $(t) - norm$. It holds:

Lemma: Let $\alpha > 0$ be fixed. The $\alpha - norm$ of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \leq \delta^{2\alpha}\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2$$

with $\delta > 0$ being arbitrary.

Remark: This inequality is in a certain sense the counterpart of the logarithmic convexity of the $\alpha - norm$, which can be reformulated in the form ($\mu, \nu > 0$, $\mu + \nu > 1$)

$$\|x\|_\beta^2 \leq \nu\varepsilon\|x\|_\gamma^2 + \mu e^{-\nu/\mu}\|x\|_\alpha^2$$

applying Young's inequality to

$$\|x\|_\beta^2 \leq (\|x\|_\alpha^2)^\mu (\|x\|_\gamma^2)^\nu \ .$$

The counterpart of lemma 4 above is

Lemma: Let $t, \delta > 0$ be fixed. To any $x \in H_0$ there is a $y = y_t(x)$ according to

i) $\|x - y\| \leq \|x\|$

ii) $\|y\|_1 \leq \delta^{-1}\|x\|$

iii) $\|x - y\|_{(t)} \leq e^{-t/\delta}\|x\|$.

Non Linear Problems

Let the problem be given by

$$F(x,u) = 0$$

with the (roughly) regularity assumptions:

i) there is a unique solution

ii) F, F_u are Lipschitz continuous.

The approximation problem is given by:

find $\varphi \in S_h \quad (F(\cdot,\varphi),\chi) = 0$ for $\chi \in S_h$.

Error analysis

Put

$$f(x) = F_u(x,u(x)) \quad \text{and} \quad \varphi = u - e$$

Then

$$(fe,\chi) = (R,\chi)$$

with a remainder term

$$R := R(e) := F(\cdot, u-e) + fe$$

resp.

$$(fe,\chi) = (fu - R(e),\chi) .$$

Let P_h denote the L_2 – projection related to $(f\cdot,\cdot) = (R,\chi)$, then

$$\varphi = P_h(u - \frac{1}{f}R(e))$$

resp.

$$e = (I - P_h)u + P_h\frac{1}{f}R(e)) =: T(e) \cdot$$

Therefore the difference $e = u - e$ is a fix point of T .

Let

$$B_{\kappa\bar{\varepsilon}} := \left\{ e \middle| \|e\|_{L_\infty} \le \kappa\bar{\varepsilon} \right\} \text{ and } \bar{\varepsilon} := \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty} .$$

With that some key properties of T are summaries in the following

Lemma:

i) There is a $\kappa > 0$ such that for $\bar{\varepsilon}$ sufficiently small, then T maps the ball $B_{\kappa\bar{\varepsilon}}$ into itself.

ii) for $\bar{\varepsilon}$ sufficiently small, T is a contradiction in $B_{\kappa\bar{\varepsilon}}$.

Proof: i) Because of P_h and f^{-1} are being bounded it holds

$$\|I - P_h\|_{L_\infty} \le c_1 \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty} = \bar{\varepsilon}$$

and

$$\left\|P_h(\frac{1}{f} R(e))\right\|_{L_\infty} \le c_2 \|R(e)\|_{L_\infty} .$$

It is
$$\|F(\cdot, u - e) + fe\|_{L_\infty} \le c_3 \|e\|_{L_\infty}^2 = c_3 \kappa^2 \bar{\varepsilon}^2$$

with c_3 being the Lipschitz constant of F_u. Therefore

$$\|T(e)\|_{L_\infty} \le c_1 \bar{\varepsilon} + c_3 c_2 \kappa^2 \bar{\varepsilon}^2 .$$

Now fixing $\kappa > c_1$ and choosing $\bar{\varepsilon}_0$ according to $\kappa = c_1 + c_3 c_2 \kappa^2 \bar{\varepsilon}_0$ gives i)

ii) it holds

$$\|T(e_1) - T(e_2)\|_{L_\infty} = \left\|P_h(\frac{1}{f}(R(e_1) - R(e_2))\right\|_{L_\infty} \le c_2 \|R(e_1) - R(e_2)\|_{L_\infty}$$

and
$$R(e_1) - R(e_2) = F(\cdot, u - e_1) - F(\cdot, u - e_2) = (F_u(\cdot, \vartheta) - F_u(u)(e_1 - e_2) .$$

With
$$F_u(\cdot, \vartheta) = F_u(\cdot, u - \vartheta e_1 - (1 - \vartheta) e_2)$$

one gets
$$\|F_u(\cdot, \vartheta) - F_u(\cdot, u)\| \le \kappa \bar{\varepsilon} c_3 .$$

Choosing
$$\bar{\varepsilon} < Min(\varepsilon_0, \frac{1}{c_2 c_3 \kappa})$$

then proves ii).

Consequence: The operator T has a unique fix-point in the ball $B_{\kappa\bar{\varepsilon}}$

From this it follows the

Theorem: The FEM admits the error estimate

$$\|u - \varphi\|_{L_\infty} \le c \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty} .$$