MATHEMATICAL ANALYSIS
OF NAVIER-STOKES EQUATIONS FOR
INCOMPRESSIBLE LIQUIDS

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1 INTRODUCTION

Let viscous incompressible liquid occupy a volume $\Omega$ of three-dimensional Euclidean space $E_3$. Introducing a Cartesian orthogonal frame $x_1, x_2, x_3$ in $E_3$, we denote a point $(x_1, x_2, x_3)$ by $x$. Let us denote the velocity and pressure of the liquid at the point $x$ and at the instant $t$ by $v(x, t)$ and $p(x, t)$, respectively, $v_i(x, t)$, $i = 1, 2, 3$, being the projections of the velocity on axes $x_i$. The scale may be chosen in such a way that the density is equal to unity. Then according to the Navier-Stokes theory the motion of the liquid caused by external forces $f(x, t)$ will be described by the following system of four equations:

\[
v_t - v \Delta v + \sum_{k=1}^{3} v_k v_x^k = -\text{grad } p + f, \tag{1}\]

\[
\text{div } v = 0. \tag{2}\]

Here $v_t$ and $v_x^k$ are derivatives of $v$ with respect to time and space coordinates respectively; $\Delta v = \sum_{k=1}^{3} v_{xx}^k$ is the Laplacian operator; $v = \text{const}$ is the kinematic viscosity coefficient; and $\text{div } v = \sum_{k=1}^{3} v_{xk}^k$. It is also assumed that the no-slip condition at the boundary $\partial \Omega$ of the domain $\Omega$ is satisfied, that is, that the liquid velocity at each point of the boundary $\partial \Omega$ coincides with the velocity of this point of the boundary. If the vessel containing the liquid does not move, this condition takes the form:

\[
v|_{\partial \Omega} = 0 \tag{3_1}\]

If the vessel does move, but in such a way that the volume $\Omega$ occupied in $E_3$ does not change (for example, if $\Omega$ is axially symmetrical), one obtains for the no-slip condition

\[
v|_{\partial \Omega} = b, \tag{3_2}\]

where $b(x, t)$ is the boundary velocity.
We restrict ourselves to these two cases. The general situation in which the vessel moves arbitrarily but retains its form may be treated in a similar way.

It is generally accepted that (1), (2) with conditions \((3_k)\) \((k = 1 \text{ or } 2)\) give a complete description of the motion of the liquid, that is, if at the moment \(t = t_0\) one knows the velocity field \(v(x, t_0)\), then \(p(x, t)\) and \(v(x, t)\) at later moments are uniquely given for \(x \in \Omega = \Omega \cup \partial \Omega\) by the solution of the system (1), (2) that satisfies the boundary conditions \((3_k)\) \((k = 1 \text{ or } 2)\). With no loss of generality let us assume that we know the velocity field at the moment \(t_0 = 0\):

\[ v_{|t=0} = a(x). \] (4)

One of the principal questions discussed in this paper (Sect. 2) is whether the system (1), (2) with the boundary conditions \((3_k)\) \((k = 1 \text{ or } 2)\) and initial condition (4) does indeed define \(v(x, t)\) and \(p(x, t)\) for \(t \geq 0\) and does so uniquely (the fields \(f, a,\) and \(b\) are known).

It may be seen immediately that if \(v(x, t)\) and \(p(x, t)\) satisfy all the above requirements they are also satisfied by \(v(x, t)\) and \(p(x, t) + c(t)\), where \(c(t)\) is an arbitrary function of time since the system (1), (2) contains only the space derivatives of \(p(x, t)\). For complete determination of \(p(x, t)\) one has to know its value in some one point \(x(t)\) at all \(t \geq 0\). In discussing the uniqueness of solution to the problem (1)–(4) we shall assume \(p(x, t)\) to contain an arbitrary summand \(c(t)\).

The second question to be discussed here (Sect. 3) is the problem of the existence of stationary flows, that is, the solvability of the boundary-value problem:

\[-\nabla p + \sum_{k=1}^{3} v_k v_{x_k} = -\nabla p + f,\]
\[\text{div } v = 0, \quad v|\partial \Omega = b.\] (5)

In this case \(f\) and \(b\) are obviously independent of \(t\).

The main results of the linearized problems are given in Sect. 4. The following paragraphs (Sections 5–7) are devoted to the stability theory for the system (1), (2), to justification of the linearization principle, to the emergence of secondary flows, and to the description of the limit \((t \to \infty)\) states. The issue of possible indeterminacy of the description of the dynamics in the Navier-Stokes theory and the search for some additional principle that enables one to determine the averaged flow is discussed in Sect. 8. In Sect. 9 we suggest some numerical experiments for powerful computers that may provide insight into the properties of the Navier-Stokes equations.

Some generalizations of the Navier-Stokes equations that satisfy Stokes’ postulates are given in Sect. 10. Such generalized equations do provide a description of the dynamics of viscous incompressible liquids that is free from indeterminacy. Finally, Sect. 11 contains a concise history of the problem.

### 2 THEOREMS OF EXISTENCE AND UNIQUENESS

The first question posed in Sect. 1 has been successfully solved for two-dimensional flows, more precisely, for the following three cases:
1. Let Ω be a cylinder with an arbitrary base. Choose the coordinate frame in $E_3$ in such a way that Ω is determined by the following conditions: $\Omega = \{ x : (x_1, x_2) \in D, x_3 \in (-\infty, \infty) \}$. Let also the vectors $f = (f_1, f_2, f_3)$ and $a = (a_1, a_2, a_3)$ from (1) and (4) be independent of $x_3$. Then the problem (1), (2), (3), (4) has a unique solution $v(x, t)$, $p(x, t)$ for $x \in \Omega, t \geq 0$, with $v$ and $p$ also independent of $x_3$. For any finite interval of time $[0, T]$ the solution is continuously dependent on $f$ and $a$. The smoothness of the solution is determined by the smoothness of the boundary and the vector-functions $f$ and $a$ as well as the order of the compatibility of the problem’s input data. As the smoothness of these data and the order of their compatibility increases so does the smoothness of the solution. These conclusions are also valid for the boundary condition (32) as well as for general boundary conditions. The above statements must be made explicit by indicating the degree of smoothness of $f, a, b,$ and $\partial D$ required for a particular smoothness of the solution. The statement on the continuity of dependence of $v, p$ on $f, a,$ and $b$ also requires detailing. All of these problems are well studied and there are estimates for $v, p$ as a functional of $f, a, b, D,$ and their deviations as a result of perturbations introduced into $f, a, b$. We shall elaborate no more on this case here. The main results with their proofs may be found in Chap. VI of Ladyzhenskaya (1970) (see also Ladyzhenskaya 1958). It should be noted that the base $D$ may be both bounded and unbounded in $E_2$, including the case of the whole $E_2$.

2. A second case, which follows, has also been studied with respect to unique solvability. Let $\Omega$ be a domain of $E_3$ obtained by rotation around the axis $x_3$ of a two-dimensional domain $D$ situated in the half-plane $\{ x : x_2 = 0, x_1 > 0 \}$ and separated from $x_3$ by a positive distance. Introduce cylindrical coordinates in $E_3(r, \theta, z)$, $z = x_3$, and let all vector-functions be represented by their cylindrical components, for example, $v = (v', \nu^\theta, v^z)$. Then, provided such components of $f$ and $a$ are $\theta$-independent, the problem (1), (2), (3), (4) has a unique solution $v = (v', \nu^\theta, v^z)$, $p$ that is also $\theta$-independent. All the assertions of case 1 are also valid for this case (see Ladyzhenskaya 1970, Chap. VI, and 1958).

3. In this case let us introduce cylindrical coordinates in $E_3$ and represent all the vector fields by their cylindrical components in a way similar to the second case. Then if $\Omega$ is the whole $E_3$ and if the cylindrical components of $f$ and $a$ are $\theta$-independent with $f^\theta = a^\theta \equiv 0$, then the problem (1), (2), (4) (i.e. the Cauchy problem) is uniquely solvable in $E_3$ for all $t \geq 0$ with $v$ and $p$ being $\theta$-independent and $v^\theta \equiv 0$. All the other assertions of case 1 are also valid here (Ladyzhenskaya 1970, Chap. VI, and 1968b).

Thus in the above three cases, which we call two-dimensional, equations (1), (2) with the boundary conditions indeed completely determine the evolution of the system. Alas, for the general three-dimensional flows we cannot be sure of this. Let us start with the description of what has been rigorously proved. First, it has been proved that for $f, a,$ and $\partial D$ with a certain smoothness and compatibility the problem (1)-(4) does have a unique solution over a certain time interval $[0, T]$. The value of $T$ depends on a quantity that may be naturally called "the generalized Reynolds number" $Re$. For $Re \to \infty$ the value of $T$ goes to zero and for $Re$ smaller than a certain value $Re^0 > 0, T$ becomes infinite (that is, the unique
solution extends over all positive times) (see Kiselev & Ladyzhenskaya 1957 and Ladyzhenskaya 1970, Chap. VI).

If the system (1), (2) has a “reasonably good” solution \( v^0(x, t) \) over some interval \([0, \tau]\), then this system is uniquely solvable over the same interval for all initial and boundary conditions for \( v \) close to \( v^0(x, 0) \) and \( v^0\rvert_{\partial\Omega} \) respectively.

The above assertion requires detailing. Several theorems of this kind are proved in Chap. VI of Ladyzhenskaya (1970). In a number of other papers there are different variants of these theorems (see Prodi 1959, 1962; Lions 1959; Serrin 1959, 1962, 1963; Kato & Fujita 1962; Solonnikov 1964b,c; Sobolevskii 1964; etc), but the main results are generally the same. No increase in the smoothness of \( f, a, \) and \( \partial\Omega \) (even their analyticity) and the order of compatibility enables one to prove the unique solvability of the problem (1)-(4) for all \( t \geq 0 \) in any functional space, provided the Reynolds number is not small. This problem focuses on one question: are there such instants when the velocity becomes infinite? If there are no such instants, that is, if \( \sup_{x \in \Omega} |v(x, t)| \) is finite for any \( t \geq 0 \) (in other words if a smooth field of velocities \( a \) under the action of a smooth field of forces \( f \) does not collapse at any instant of time, that is, if \( |v(x, t)| \) does not become infinite) then (1) and (2) with the no-slip conditions are sufficient for complete determination of the evolution of an initial velocity field.

There are finer criteria for unique solvability of the problem (1)-(4) “in the large” (that is, for any time interval and any Reynolds numbers) (see theorem 12', Sect. 4, Chap. VI of Ladyzhenskaya 1970). Thus, for example, in the case of a bounded \( \Omega \) it is sufficient to know that for all possible smooth solutions \( v(x, t, \lambda) \) of the problems

\[
v_t - v \Delta v + \lambda \sum_{k=1}^{3} v_k v_{x_k} = -\text{grad} p + f, \quad \text{div } v = 0, \quad v\rvert_{\partial\Omega} = 0, \quad v\rvert_{t=0} = a, \quad \lambda \in [0, 1],
\]

one of the norms

\[
\left( \int_{\Omega} |v(x, t)|^q \, dx \right)^{1/q}, \quad q > 3, \quad \text{or} \quad \left( \int_{\Omega} \sum_{i,k=1}^{3} v_{x_k}^2(x, t, \lambda) \, dx \right)^{1/2}
\]

for all \( t \in [0, T] \) does not exceed a constant \( C(T) \) that must depend only on norms of \( f \) and \( a \), the value of \( T \), and on characteristics of \( \Omega \). These criteria of unique solvability of the problem (1)-(4) allow the values of \( |v_{x_k}(x, t)| \) and even of \( |v(x, t)| \) to become infinite for some points of time-space but in such a way that one of the above mentioned norms stays finite for all \( t \in [0, T] \). If one of these criteria is satisfied, it insures the unique solvability of the problem, and its solution will be finite and smooth everywhere if only \( f, a, \) and \( \partial\Omega \) are smooth and the necessary conditions of compatibility are satisfied.

The majority of hydrodynamicians believe that the presence of the term \( -v \Delta v \), responsible for the viscosity in (1), leads to smoothing of the flow. However, no one has proved it as yet and we think that for general three-dimensional flows and large Reynolds numbers this does not hold. In Sect. 8 we return to the
discussion of the sufficiency of (1) and (2) and the conditions (3k) for the determination of the evolution of the initial flows $a(x)$. We now present the results relevant to problem (5).

3 ON THE SOLVABILITY OF STATIONARY PROBLEMS

This section is devoted to the question of the solvability of (5) when the forces $f$ and boundary flow $b$ are time-independent. Let for the time being $\Omega$ be a bounded domain of $E_3$. Its boundary $\partial \Omega$ may consist of a number of separate closed surfaces $S_i$, $i = 1, \ldots, N$ ($\partial \Omega = \bigcup_{i=1}^{N} S_i$). Let the field $b$, known on all $S_i$, $i = 1, \ldots, N$, satisfy the conditions

$$ j_i \equiv \int_{S_i} (b, n) \, ds = 0, \quad i = 1, \ldots, N. \tag{6} $$

Here $n$ is a unit normal to $S_i$ directed out of $\Omega$. It should be noted that the equality $\sum_{i=1}^{N} j_i = 0$ is a necessary condition of the solvability of (5) since

$$ \int_{\Omega} \text{div} \, v \, dx = 0 = \int_{\partial \Omega} (b, n) \, ds = \sum_{i=1}^{N} j_i. $$

The conditions (6) are definitely satisfied in the problem of flow about $N - 1$ bodies inside a certain enveloping surface. It has been proved that (5) when the conditions (6) are satisfied has at least one solution $v, p$. The smoothness of any solution to this problem at an arbitrary point $x^0$ inside $\Omega$ is determined only by the smoothness of the field $f$ in the vicinity of this point while its behaviour at a boundary point $x^0$ depends also on the smoothness of both $b$ and $\partial \Omega$ in the vicinity of $x^0$. Here we also skip detailed formulations but point out that the boundary surfaces $S_i$ need not be smooth (see Leray 1933, Ladyzhenskaya 1959, 1970, Vorovich & Yudovich 1961, and H. Fujita 1961).

Solutions guaranteed by these theorems may naturally be called laminar since $v(x)$ and $p(x)$ are smooth within $\Omega$ provided $f(x)$ is smooth. They exist for arbitrarily large Reynolds numbers. This fact seemed to contradict experiments in which laminar flows disappear for sufficiently large Reynolds numbers. In light of the above the fact that laminar flows for large Reynolds numbers are not observed should not be attributed not to their absence but to their instability. The conditions (6) may be softened: the right-hand sides need not be zero. However, to prove the solvability of (5) one has to require that they are "sufficiently small." It is not known whether this restriction may be removed and replaced by the only necessary condition $\sum_{i=1}^{N} j_i = 0$. For plane problems in bounded domains analogous results have been obtained.

Another important question is that of the number of possible solutions to (5). For "small" Reynolds numbers the solution is unique. It seems natural to expect the appearance of other and even infinitely many solutions for large Reynolds numbers. The examples of nonlinear elliptic boundary problems make such an expectation reasonable. However, no example of a bounded domain $\Omega$ with $\partial \Omega$ consisting of a single component is yet known for which (5) has even two different
solutions for any \( f \) and \( b \). There are only two examples of non-uniqueness of the solutions to (5). One is that of a plane problem outside the unit circle (see the preface to the second American edition of Ladyzhenskaya 1970). In that case a continuum set of solutions to (5) with \( f = 0 \), depending on one parameter, satisfy the same boundary condition and go to zero with \( |x| \to \infty \). The other example (see Yudovich 1967) is that of a plane annulus \( 0 < \rho_1 \leq (x_1^2 + x_2^2)^{1/2} \leq \rho_2 \), within which \( f \) and \( b \) for which at least two different solutions exist have been indicated. The conditions (6) are not satisfied in both these examples.

Consider now unbounded domains. Attention has been given mostly to the problem of \( N \) bounded immobile rigid bodies \( \Omega_i, \ i = 1, \ldots, N \), in \( E^3 \) situated in a flow reaching a constant velocity \( v^\infty \) as \( |x| \to \infty \) (though there are a number of results for more general formulations of the problem). This problem consists of finding functions \( v(x), p(x) \) which satisfy in \( \Omega = E_3 \setminus \cup_{i=1}^{N} \Omega_i \) the following system of equations:

\[
\begin{align*}
- \nu \Delta v + \sum_{k=1}^{3} v_k v_{x_k} &= - \text{grad} \ p, \\
\text{div} \ v &= 0,
\end{align*}
\]

(7)
together with the boundary conditions

\[
v \big|_{\partial \Omega} = 0
\]

(8)
at \( \partial \Omega = \cup_{i=1}^{N} \partial \Omega_i \) and

\[
\lim_{|x| \to \infty} v(x) = v^\infty = \text{const}
\]

(9)
at infinity. The proof of the existence of at least one solution to it for which the Dirichlet integral

\[
\int_{\Omega} \sum_{i,k=1}^{3} v_{x_i}^2 \, dx
\]

is finite is comparatively simple (Leray 1933, Ladyzhenskaya 1959 and 1970, Chap. V). This solution is infinitely differentiable inside \( \Omega \) and \( v(x) \) approaches \( v^\infty \) as \( |x| \to \infty \) uniformly with respect to all directions. Its smoothness in the boundary points depends only on the smoothness of the part of the boundary adjacent to these points (these results do not change if (8) is replaced by

\[
v \big|_{\partial \Omega} = b
\]

(10)
provided \( \int_{\partial \Omega_i} (b, n) \, ds = 0, \ i = 1, \ldots, N \).

However, for such a solution no other characteristics of the behavior of \( v(x) \) for \( |x| \to \infty \) have been obtained: in particular it is not known how rapidly \( v(x) \) approaches \( v^\infty \) when \( x \) goes to \( \infty \) along various directions. It was natural to expect that in the case of a single body \( \Omega_i \) there would be a “parabolic tail” stretching behind it in the direction of \( v^\infty \), and that \( v(x) \) approaches \( v^\infty \) much faster outside this tail than within it. These and other important aspects of the behavior of the field \( v(x) \) for \( |x| \to \infty \) were the subject of studies by Finn (1965).
He has proved the existence of the unique classical solution to (7)-(9) in the case \( N = 1 \), provided \(|v|^\infty\) is sufficiently small, and has studied its asymptotic behavior through the development of the classical methods of the theory of the potential. He has done it also for the inhomogeneous boundary condition (10), provided \(|\mathbf{b}^1|\) and \(|v|^\infty\) are sufficiently small. For arbitrary \( \mathbf{b} \) and \( v \), these questions still left open.

For plane-parallel flows the flow problems proved to be more difficult. The analysis of the well-known Stokes paradox for the two-dimensional linearized equation \( \nu \Delta v = \text{grad} \ p \) \( \{ \text{here } \Delta v = \sum_{k=1}^{2} v_{x_k x_k} \text{ and } v = [v_1(x_1, x_2), v_2(x_1, x_2)] \} \) have shown that (9) must be replaced by the requirement of finiteness of either \(|v(x)| \) at \( |x| \rightarrow \infty \) or the integral
\[
\int_\Omega \sum_{i,k=1}^{2} v_{x_i k}^2 \, dx.
\]
This has made it unclear whether a condition of the same type as (9) should be required at infinity for complete equations such as (7). Finn & Smith (1967) have shown that conditions of the type (9) are necessary and have proved the solvability of (7)-(9) for \( N = 1 \) and plane-parallel flows provided \( v \) is sufficiently small.

### 4 LINEARIZED PROBLEMS

The problems discussed in Sections 2 and 3 were preceded by the analysis of linearized Navier-Stokes equations and, first of all, their Stokes linearizations:

\[
\mathbf{u}, - \nu \Delta \mathbf{u} = -\text{grad} \ p + \mathbf{f}(x, t), \quad \text{div} \ \mathbf{u} = 0
\]  
(11)

and

\[
-\nu \Delta \mathbf{u} = -\text{grad} \ p + \mathbf{f}(x), \quad \text{div} \ \mathbf{u} = 0.
\]
(12)

The same questions were studied for (11) and (12) as for the complete Navier-Stokes system. Moreover, for the case of steady flow the spectral problem

\[
v \Delta \mathbf{u} - \text{grad} \ q = \lambda \mathbf{u}, \quad \text{div} \ \mathbf{u} = 0, \quad \mathbf{u}|_{\partial \Omega} = 0,
\]
(13)

has been investigated. Here \( \lambda \) is the spectral (possibly complex) parameter. It is necessary to find those values of \( \lambda \) that correspond to non-trivial (that is, other than identically zero) solutions. Such values of \( \lambda \) constitute the spectrum of the problem and corresponding solutions are called eigen-elements. System (11) has been extensively analyzed. The results resemble those obtained for the heat equation. But there are also important differences. For example, there is the so-called maximum principle for solutions of the heat equation, which is not valid for system (11) and no analogues to it have been found. Many results for system (12) resemble those of the Poisson equation, \( \Delta \mathbf{u} = \mathbf{f} \), and the properties of the spectral problem (13) are similar to those of \( \Delta \mathbf{u} = \lambda \mathbf{u} \), \( \mathbf{u}|_{\partial \Omega} = 0 \) for the \( \Delta \) operator. In particular, for the bounded region \( \Omega \) the spectrum of problem (13) consists of a countable number of
negative values \( \{ \lambda_k \} \) which may be ordered: \( 0 > \lambda_1 \geq \lambda_2 \geq \ldots \). Each \( \lambda_k \) is of finite multiplicity and \( \lambda_k \) goes to minus infinity as \( k \to \infty \). We shall not include here results for all the above mentioned problems, which may be found in Ladyzhenskaya (1970). Part of them have been obtained by F. K. G. Odqvist (1930). The rest date back to the 1950s and early 1960s (Ladyzhenskaya 1970; Golovkin & Solonnikov 1961; Solonnikov 1960, 1964a,b,c, 1966; Cattabriga 1961; Kato & Fujita 1962 etc).

The nonlinear problems required the study of other linearizations of the Navier-Stokes equations:

\[
\begin{align*}
u \Delta u &+ \sum_{k=1}^{3} c_k(x,t)u_{x_k} + C(x,t)u = -\text{grad } q + f(x,t), \\
\text{div } u &= 0,
\end{align*}
\]

(14)

\[
\begin{align*}-v \Delta u + \sum_{k=1}^{3} c_k(x)u_{x_k} + C(x)u &= -\text{grad } q + f(x), \\
\text{div } u &= 0
\end{align*}
\]

(15)

and

\[
\begin{align*}v \Delta u - \sum_{k=1}^{3} c_k(x)u_{x_k} - C(x)u - \text{grad } q &= \lambda u, \\
\text{div } u &= 0, \quad u|_{\partial \Omega} = 0.
\end{align*}
\]

(16)

Here \( c = (c_1, c_2, c_3) \) is a known vector-function and \( C \) a known matrix with variable elements in the general case. One comes across such equations for example in the theory of stability (see Sections 5 and 6). For (14) the question of unique solvability of initial-boundary-value problems (roughly speaking they always are uniquely solvable) has been rather extensively studied and quite useful estimates both for the Green function and for the solutions have been obtained. This was carried out for both bounded and unbounded domains \( \Omega \). The boundary conditions of the type \( u|_{\partial \Omega} = b(x,t) \) were used for the most part, but periodic boundary conditions (either on the whole or on some part of the boundary) may be analogously treated.

Problem (15) has been investigated mostly for bounded domains though there are some results for unbounded ones. It may be said that the final results are similar to those for a single scalar elliptic equation of the second order. The spectral problem (16) has been studied considerably less, although it is quite important for the theory of stability of Navier-Stokes equations. For arbitrary bounded domain \( \Omega \) the only thing known about it is that its spectrum consists of an infinite number of eigenvalues \( \{ \lambda_k \} \) which may be ordered so that \( |\lambda_1| \leq |\lambda_2| \leq \ldots \) and \( |\lambda_k| \to \infty \) when \( k \to \infty \) with each \( \lambda_k \) having a finite rank. The eigenvalues may be complex and are situated within the parabola: \( \{ \lambda = \lambda' + i\lambda'', \lambda' < C_1 + C_2 \lambda'^2 \} \). The constants \( C_1 \) and \( C_2 \) may be expressed in terms of the coefficients \( c_k(x) \) and \( C(x) \) and \( \Omega \). However, such a description of the spectrum ignores the specifics of concrete coefficients and turns out to be too crude for the analysis of stability and bifurcation problems. The spectral problems corresponding to all known flows
studied in the hydrodynamical theory of stability turn out not to be selfadjoint. Functional analysis contains a comparatively large number of results pertinent to selfadjoint problems while the few results obtained for the nonselfadjoint ones are of little value for problems of the type (16). For example, in these latter problems no criteria are known for the existence of at least one real eigenvalue or for the minimum-real-part eigenvalue to be either real or imaginary. Lack of results of this kind makes one invest special methods for each particular case of \( c_k(x), k = 1, 2, 3, \) and \( C(x), \) some of which are to be found in Sect. 6. It should be noted that the main difficulties in the study of the properties of (14) and (15) mentioned above are mainly overcome already at the first stage—in the investigation of these properties for systems (11) and (12) respectively. The presence of “minor terms” in (14) does not affect such properties as the unique solvability of initial-boundary-value problems for (14) or qualitative properties of its solutions (provided, of course, \( c \) and \( C \) are “not too discontinuous”). Such terms even cannot “spoil” the Fredholm solvability of boundary problems for (15) within a bounded domain \( \Omega \). However, \( c \) and \( C \) become essential for the asymptotics of solutions at large \( t \) or \( |x| \) (if \( \Omega \) is unbounded). Results obtained concerning these matters are rather sparse (see Ladyzhenskaya 1970, Finn & Smith 1967, V. A. Solonnikov 1964b,c, 1973, and Panich 1962).

5 STABILITY PROBLEMS. THE PRINCIPLE OF LINEARIZATION

This section deals with three-dimensional flows within a bounded region with only the initial field of velocities being varied. The boundary values of \( v \) are taken to be known and fixed. The cases of periodic boundary conditions (either on the whole or on some part of the boundary) and changes of the flow caused by varying \( f \) or \( b \) may be treated in a similar way. Everything we assert for three-dimensional flows is also valid in two dimensions. The stability problems discussed in this paragraph have not been considered for unbounded domains, though in a number of important cases they do yield to analysis. Let system (1), (2) possess within \( \Omega \) a solution \( v(x, t), p(x, t) \) determined for all \( t \geq 0 \). We concern ourselves with the question whether the field \( v(x, t) \) is stable with respect to small perturbations of \( v(x, 0) \). Denote as \( v'(x, t), p'(x, t) \) the solution of the same system (1), (2) with boundary conditions \( v|_{\partial \Omega} = v'|_{\partial \Omega} \) and initial condition \( v(x, 0) = v(x, 0) + a(x) \). If it does exist, the difference \( u(x, t) = v'(x, t) - v(x, t) \) is the solution to the problem:

\[
\begin{align*}
\nu_t - \nu \Delta u + & \sum_{k=1}^{3} u_k u_{x_k} + \sum_{k=1}^{3} (v_k u_{x_k} + u_k v_{x_k}) = -\text{grad } q, \\
\text{div } u = 0, & \quad u|_{\partial \Omega} = 0, \quad u|_{t=0} = a.
\end{align*}
\]

(17)

Consider the corresponding linearized problem:

\[
\begin{align*}
w_t - \nu \Delta w + & \sum_{k=1}^{3} (v_k w_{x_k} + w_k v_{x_k}) = -\text{grad } q, \\
\text{div } w = 0, & \quad w|_{\partial \Omega} = 0, \quad w|_{t=0} = a.
\end{align*}
\]

(18)
It has been proved that (18) is uniquely solvable over all \( t \geq 0 \) and any \( a(x) \) (here as in the rest of this paper we indicate the degree of smoothness of neither the data nor the solutions under consideration. In this section the most beautiful results are those obtained in the class of solutions possessing the so-called generalized derivatives that enter (1), (2) and are square summable over \((x, t) \in \Omega \times (0, T)\) for every \( T < \infty \).

Assume that for the studied field \( \nu(x, t) \) the following holds

\[
\sum_{i,k=1}^{3} v_{ikx}(x, t)^2 + \nu(x, t)^2 \leq L_{\Omega}(x, t) < \infty.
\]

Then, provided all solutions \( \nu(x, t) \) to (18) that correspond to various \( a(x) \) decay exponentially as \( t \to \infty \), (17) is uniquely solvable over \( t \geq 0 \) for any sufficiently small \( a(x) \), and its solutions also decay exponentially as \( t \to \infty \). Thus, such behavior of solutions to (18) insures stability of the considered solution \( \nu(x, t) \) [including the existence of solutions \( \nu'(x, t) \) for all \( t \geq 0 \) with \( \nu'(x, 0) \) close to \( \nu(x, 0) \)].

However, the stability criterion for \( \nu(x, t) \) presented above (Ladyzhenskaya & Solonnikov 1973) practically cannot be tested. Criteria of stability for the cases of steady \( \nu(x) \) and periodic \( \nu(x, t) \) that follow from it are somewhat simpler. For the steady solutions it is formulated in the following way. Consider the spectral problem (18) that corresponds to the solution \( \nu(x) \):

\[
v \Delta z - \sum_{i,k=1}^{3} (v_{ik} z_{x_i} + z_{x_i} v_{ik}) - \text{grad } g' = \lambda z,
\]

\[
\text{div } z = 0, \quad z|_{\partial \Omega} = 0.
\]

If all the spectrum \( \{\lambda_k\} \) of this problem is in the left half-plane (i.e. all \( \text{Re } \lambda_k \leq c < 0 \)), solution \( \nu(x) \) is stable. This stability criterion, well known for the systems of ordinary differential equations, has by now been proved for the Navier-Stokes system (Prodi 1962, Sattinger 1970, and Yudovich 1965b). It has also been proved that if the real part of at least one of the eigenvalues of problem (19) is positive, solution \( \nu(x) \) is unstable (Sattinger 1970, Yudovich 1965b). Moreover, if the spectrum \( \{\lambda_k\} \) is situated on both sides of the imaginary axis and \( \text{Re } \lambda_k = 0 \) for no \( \lambda_k \), then in a small vicinity \( K_\varepsilon \) of the “point” \( \nu(x) \) there are two invariant manifolds \( \mathfrak{M}_1 \) and \( \mathfrak{M}_2 \) passing through the “point” \( \nu(x) \) such that the trajectories \( \nu'(x, t) \) (solutions), originating (at the initial moment \( t = 0 \)) from the points of manifold \( \mathfrak{M}_1 \) until a certain moment \( t' > 0 \) [which depends on \( \nu(x, 0) \)] stay within \( \mathfrak{M}_1 \) and then leave the vicinity \( K_\varepsilon \). These trajectories exist for all \( t < 0 \) and go to \( \nu(x) \) as \( t \to -\infty \). Trajectories originating on \( \mathfrak{M}_2 \) stay within it and go to \( \nu(x) \) as \( t \to \infty \). And finally, trajectories originating on other “points” of \( K_\varepsilon \) leave it after a while.

Analogous results were obtained also for periodic solutions \( \nu(x, t) \), both induced and auto-oscillating. All this has been proved not only for the Navier-Stokes equations (see Yudovich 1970a, 1970b) but also for a number of problems of heat convection and magnetic hydrodynamics of viscous incompressible liquids (Ladyzhenskaya & Solonnikov 1973). From a purely mathematical point of view these results look fine but they hardly lend themselves to test, even in the simpler
case of steady solutions $v(x)$. This is connected to the fact that the spectral problem (19) has hardly been studied at all, as has been pointed out in Sect. 4. There are only disconnected results relevant to particular choices of $\Omega$ and flows $v(x)$ within them. The next section is devoted to a concise description of these results.

6 THE STUDY OF STABILITY OF SOME PARTICULAR FLOWS. SECONDARY FLOWS

6.1 Kolmogorov’s Flow

Consider a plane flow $v = (yv^{-1}\sin\gamma, 0)$, $p = $ const in the strip $\Pi = \{(x, y): -\infty < x < \infty, -\pi \leq y \leq \pi\}$ corresponding to force $f = (\gamma \sin y, 0)$ and study those perturbations of it [or rather solutions of spectral problem (19)] which are $2\pi/\alpha$-periodic functions of $x$. Here $\alpha$ as well as $\gamma$ and $v$ are variable parameters of the problem. The eigenvalues corresponding to these solutions were found to move from the left half plane $\{\lambda\}$ to the right one as $\gamma/v^2$ increased and $\alpha < 1$. This insures instability of the flow $v = (yv^{-1}\sin\gamma, 0)$ for sufficiently large $\gamma/v^2$. This fact had been predicted by Kolmogorov and subsequently proved by Meshalkin & Sinai (1961) (see also Yudovich 1965a).

6.2 Plane-Parallel Flow of Couette (Linear Profile)

It was believed that the linear profile flow is probably stable under periodic perturbations of any period $\alpha$ and for all Reynolds numbers $Re$. W. Wasow (1953), Yu. B. Ponomarenko (1968), L. A. Dikii (1964), and others have proved this for various regions of $(\alpha, Re)$ values. The rest of the parametric quarterplane $E^+ = \{(\alpha, Re): \alpha > 0, Re > 0\}$ was recently investigated by V. A. Romanov (1971) and the solution there proved to be stable. Thus it has been shown that the linear-profile flow is stable for any $\alpha$ and all $Re > 0$.

6.3 Poiseuille Flow Inside a Plane Channel (Symmetric Parabolic Profile)

Heisenberg has studied this profile and concluded that it loses stability for certain values of parameters $\alpha$ and $Re$. However, his reasoning was not rigorous and is opposed, along with his final conclusions, by a number of scientists. The difficulties with this profile are connected with the fact that in regions where either $\alpha$ or $Re$ are large the profile is stable. This makes necessary the analysis of “intermediate” regions of values of the parameters $\alpha$, $Re$. This has been done by Krylov (1966), who indeed found such areas in which Poiseuille flow becomes unstable. Previous (non-rigorous) results of Lin led to the same conclusions.

6.4 Couette Flow Between Two Rotating Cylinders

Couette flow has been the subject of widely known researches by G. I. Taylor. He demonstrated that this flow becomes unstable for certain values of the parameters and new flows emerge for near-by parameter values. From a mathematical point of view these conclusions needed proof. A. L. Krylov (1963) considered the case of cylinders rotating in the same direction and proved for this case that loss of stability is indeed possible. Yudovich (1966) has also done this somewhat more
simply. The appearance of a secondary stationary flow branching from Couette flow has been proved in the papers of Velte (1964, 1966), Yudovich (1966), Ivanilov & Yakovlev (1966), and Kirchgässner & Sörger (1969). A number of methods of analysis were employed including that of Lyapunov-Schmidt. It should be mentioned that some results are still somewhat conditional since not all of the assumptions used are proved. The only exception is the case of small gap between cylinders analyzed by Yudovich.

6.5 On the Studies of Other Flows

The only flow for which the spectral problem (19) turned out to be symmetrizable was indicated by Yudovich (1967): \( v = (v_r, v_\theta, v_z), \ v_r = 0, \ v_\theta = yr_\theta, \ v_z = 0 \) in a bounded domain of torus-type (with \( f \neq 0 \)). Thanks to this, it proved possible to carry out a detailed analysis of the spectral problem and to demonstrate rigorously not only the appearance of instability, but also the emergence of a secondary stationary flow. The stability of the latter was also investigated (under some assumptions).

Instability of Couette flow in a round pipe could not be found within the class of solutions independent of the rotation angle, but it is likely to exist in a wider class of solutions (see Gill 1965). In a number of recent papers Iooss (1971a,b,c, 1972), Sattinger (1971), Joseph & Sattinger (1972), and Yudovich (1971, 1972) studied the emergence of secondary stationary and periodic solutions close to the stationary one (or rather to the family of solutions depending on a numerical parameter) defined over an arbitrary domain \( \Omega \). Secondary stationary flows are obtained by Lyapunov-Schmidt's method. In order to obtain secondary periodic solutions, various series representations are suggested. In the cited papers a number of criteria are given which are sufficient to tell one situation from the other. These criteria look simple in comparison with the problem itself, but it is quite tedious to check them for particular flows.

7 ON THE LIMIT-STATES FOR PROBLEM (1), (2), (3)

However important the methods of Sect. 6 for the study of behavior of particular flows are, their limitations are nevertheless obvious. They enable us at best to find solutions emerging in the vicinity of the one considered and there is nothing to insure that as the Reynolds number increases other solutions which are far from the one we study will not emerge with better stability than the secondary flows we have discussed in Sect. 6. We think that the following approach to the study of limit-states for arbitrary Reynolds numbers is interesting. Unfortunately, this approach has proved fruitful as yet only for two-dimensional flows since only for this case have the unique solvability for all \( t \geq 0 \) and the correctness of initial-boundary problems of (1)–(4) been proved for arbitrary Reynolds numbers. Let us present this approach here for the following problem:

\[
\begin{align*}
&v_t - v \Delta v + \sum_{k=1}^{2} \nu_k \varepsilon_{x_k} = -\text{grad} \ p + f(x), \\
&\text{div} \ v = 0, \quad v_{\mid \partial \Omega} = 0, \quad v_{\mid r = 0} = a(x),
\end{align*}
\]
taking $x = (x_1, x_2)$ from a bounded domain $D$ in Euclidean space $E_2$ and $v, f, a$ as two-component vector-functions. The unique solvability of this problem has been proved for all $t \geq 0$ for any $a(x)$ belonging to some subspace $\tilde{J}(D)$ of Hilbert space $L_2(D)$. The elements of $L_2(D)$ are vector-functions $u(x)$, square-summable over domain $D$, with the scalar product defined as

$$(u, v) = \sum_{k=1}^{2} u_k(x)v_k(x) \, dx$$

and the norm $\|u\| = (u, u)^{1/2}$. The subspace $\tilde{J}(D)$ is defined as the closure in the norm of $L_2(D)$ of the set of all smooth solenoidal vector-functions $u(x)$ which become zero at the boundary $\partial D$. The vector-function $f(x)$ as well as the boundary $\partial D$ are taken here to be smooth and with no loss of generality $f \in \tilde{J}(D)$. They are also fixed in the following considerations. Call the elements $a(x)$ of subspace $\tilde{J}(D)$ “points” and the solution to (20), $v(x, t)$, a “trajectory” originating at the moment $t = 0$ from point $a(x)$; denote such a solution by $V_r a$.

Consider a sphere $k_R = \{a: \|a\| \leq R\}$ in space $\tilde{J}(D)$ and let trajectories $V_r a$ originate from each point of this sphere. It turns out that if $R$ is taken to be $\geq R_0 = \lambda_1^{-1} \|f\|$, where $\lambda_1$ is the first eigenvalue of (13), the trajectories $V_r a$ originating from points $a$ within the sphere $k_R$ do not leave this sphere at any moment $t \geq 0$. Let us follow the set $k_R(t)$ obtained from $k_R$ by the application of the nonlinear operator $V_r$. There holds the inclusion: $k_R(t_2) \subset k_R(t_1)$ for $t_2 > t_1$, and the set $k_R(t)$ for any $t > 0$ is “considerably smaller” than the sphere $k_R$. The elements (points) of $k_R(t)$, $t > 0$, are “very smooth” vector-functions. Consider the intersection $M_R = \bigcap_{t \geq 0} k_R(t)$. The elements of $M_R$ are the velocity fields to be observed in the flow after an infinite interval of time, that is, exactly the thing we wish to know. The experiment is staged by starting with various (“random”) fields $a(x) \in k_R$ and learning what they become after a “long” time. It is natural to assume that viscosity will make the flow “forget” its past and develop under the action of permanently acting factors: the force $f(x)$ and the form of domain $D$. Had the system been free from nonlinear terms $v_k v_{x_k}$, the only limit-state would have been $u(x)$—the solution of the stationary problem

$$-\nu \Delta u = -\nabla q + f(x), \quad \text{div } u = 0, \quad u|_{\partial D} = 0,$$  

(21)

no matter what $a$ we had started with. Indeed, if one decomposes $a(x)$ and $f(x)$ into series with respect to eigenfunctions $\{\varphi^k(x)\}$ of problem (13):

$$a(x) = \sum_{k=1}^{\infty} a^k \varphi^k(x), \quad f(x) = \sum_{k=1}^{\infty} f^k \varphi^k(x),$$

the solution $u(x, t)$ to the linearized problem (20) may be represented by the following series:

$$u(x, t) = \sum_{k=1}^{\infty} a^k e^{\lambda_k t} \varphi^k(x) + \sum_{k=1}^{\infty} f^k (e^{\lambda_k t} - 1) \lambda_k^{-1} \varphi^k(x)$$  

(22)

and hence

$$\lim_{t \to \infty} u(x, t) = -\sum_{k=1}^{\infty} f^k \lambda_k^{-1} \varphi^k(x).$$
It may be seen from (22) that various harmonics [terms in (22) correspond
to
different \( \phi^k \)] do not interfere. This effect will obviously disappear in the case of
nonlinear problem (20). With passing of time energy will be exchanged between
harmonics, but it is natural to assume that the influence of high harmonics will
decrease with time. Let us describe the results proved for the limit set \( \mathcal{M}_R \). First,
\( \mathcal{M}_R = \mathcal{M}_{R_0} \) for all \( R \geq R_0 \). The set \( \mathcal{M}_R \) (as well as all sets \( k(t) \) for \( t > 0 \)) is
compact within \( \hat{J}(D) \). It consists of those and only those elements \( a(x) \) of space
\( J(D) \) for which (20) is uniquely solvable both for \( t \in [0, \infty) \) and for \( t \in (-\infty, 0) \). The
set \( \mathcal{M}_R \) is an invariant of (20), that is, if \( a \in \mathcal{M}_R \) then all trajectories \( V_t, a \) at any
t \( t \in (-\infty, \infty) \) belong to \( \mathcal{M}_R \). The problem (20) defines a dynamical system \( V_t \) over
\( \mathcal{M}_R \). In particular, trajectories \( V_t, a, V_t, a' \) starting at different points \( a \) and \( a' \) cross
nowhere, that is, \( V_t, a \neq V_t, a' \) for all \( t \) and \( V_t, a \) depends on \( a \) continuously over any
finite interval of time \( [-T; T] \). Moreover, the dynamical system \( V_t \) “behaves” on
\( \mathcal{M}_R \) as a finite-dimensional one, that is, two numbers \( |f| \) and \( \lambda_1 \) define a number
\( n \) such that if one considers a finite-dimensional linear subspace \( L^{(n)} \) of space
\( \hat{J}(D) \) spanned by the first \( n \) eigenfunctions \( \{\phi^k\} \), \( k = 1, \ldots, n \), of the spectral problem
(13), and if one denotes by \( P_n \) the orthogonal operator projecting \( \hat{J}(D) \) onto \( L^{(n)} \),
then projection \( P_n V_t, a \) of any complete trajectory \( V_t, a[t \in (-\infty, \infty)] \) belonging to
\( \mathcal{M}_R \) completely defines the trajectory \( V_t, a \) itself. Also if \( P_n V_t, a \) is a time-independent,
\( \omega \)-periodic or almost periodic function of \( t \), so is \( V_t, a \).

The set \( \mathcal{M}_R \) definitely contains all stationary, periodic, and almost periodic
solutions to (20). According to Bogolyubov-Krylov theory there exist invariant
measures \( \mu \) that may be determined according to procedures described by these
authors. The structure of set \( \mathcal{M}_R \) essentially depends upon the Reynolds number.
In particular, for small Reynolds numbers \( \mathcal{M}_R \) consists of one point, the only
stationary solution to (20). We think that \( \mathcal{M}_R \) is one of the main objects worthy
of comprehensive study.

Results listed above come from the author’s paper (Ladyzhenskaya 1972)
written under the influence of E. Hopf’s study (1948). Hopf constructed a certain
model system of equations for which he proved that its limit-set is a manifold, the
dimensionality of which increases with increase of a certain parameter that may
naturally be called its “Reynolds number.” This model problem, however, lacks
interaction between harmonics and thus does not reflect one of the most important
aspects of nonlinear problem (20). For the latter the set \( \mathcal{M}_R \) will not in the general
case be a manifold in space \( \hat{J}(D) \).

8  ON THE POSSIBILITY OF INDETERMINACY IN THE
DESCRIPTION OF THE DYNAMICS OF THE
NAVIER-STOKES THEORY

The list of results obtained on the unique solvability of initial-boundary value
problems for system (1), (2) was presented in Sect. 2. It was stated there that
despite multiple attempts to prove that the general three-dimensional problem is
uniquely solvable this is still an open question. This question has two sides: first,
for which classes of solutions does the uniqueness theorem hold and second, which
solutions to problems (1)-(4) exist. The uniqueness theorem definitely holds in the class of smooth solutions. Moreover, it remains valid also within wide classes $L_{q,r}(Q_T)$ of discontinuous solutions for which components of $v(x,t)$ are such that integrals $\int_0^T \int_Q |v_i(x,t)|^q \, dx \, dt$ are finite with a pair of parameters $(q,r)$ satisfying conditions (a) $1/r + 3/2q = 1/2, r \in [2, \infty), q = (3, \infty]$ or (b) $q > 3, r = \infty$ (that is, condition $\int_0^T |v_i(x,t)|^q \, dx < \text{const}, t \in [0, T]$ with $q > 3$ must be satisfied). Such solutions satisfy the requirements of (1)-(4) in some generalized sense (see Ladyzhenskaya 1970, Sect. 2, Chap. VI; Prodi 1959, Serrin 1963, and Ladyzhenskaya 1967a). We think that this result is exact, that is, that a singularity in $v(x,t)$ stronger than that allowed by conditions (a) or (b) would ruin the validity of the uniqueness theorem. To prove this we have constructed an example (see Ladyzhenskaya 1969 and 1970, Sect. 7, Chap. VI) in which for system (1), (2) two different solutions satisfying the same boundary and initial conditions (with the same right-hand part) were indicated. Both solutions belong to $L_{q,r}(Q_T)$, but with $(q,r)$ smaller (by any positive $\varepsilon$) than those required by conditions (a) and (b). True, this example was constructed for boundary conditions differing somewhat from (3), but these other conditions from the mathematical point of view are no worse than the no-slip condition. The same kind of existence and uniqueness theorems hold for these conditions as for the no-slip ones.

Consider now existence theorems. In the general case it was proved (Hopf 1951) that at least one solution $v(x,t), p(x,t)$ of the problem (1), (2), (3), (4) for which integral

$$\int_0^T \int_\Omega \sum_{i=1}^3 v_{x_i}^2 \, dx \, dt$$

and

$$\operatorname{vrai} \max_{0 \leq t \leq T} \int_\Omega |v(x,t)|^2 \, dx$$

are finite exists (here and below $T$ is an arbitrary finite interval of time). This solution has generalized derivatives $v_i, v_{x_i x_i}, p_x$, summable over $Q_T = \Omega \times (0, T)$ with the power of 5/4, and for almost all $(x, t)$ satisfies system (1), (2) (see Golovkin & Ladyzhenskaya 1960 and Solonnikov 1964b,c). All these properties of $v(x,t)$ nevertheless do not guarantee that conditions (a) and (b) of the uniqueness theorem are satisfied. Provided $f(x,t), a(x,t), \partial \Omega$ are sufficiently smooth, this solution is also smooth over some interval of time $[0, \tau]$ and hence unique there. But one can not exclude the possibility that at some moment this smoothness will be destroyed [in spite of the smooth $f(x,t)$] to such an extent that $v(x,t)$ will no longer satisfy conditions (a) and (b) of the uniqueness theorem. At such catastrophic moments the solution may branch. Since we are trying to give a deterministic description of the process it should be understood which of the branches is chosen by the system. Such a choice requires another postulated law. We think that such a branching of the solution is possible in the Navier-Stokes equations and that this additional postulate is indeed necessary. This situation is familiar from the theory of nonviscous compressible fluids. There equations of
motion are complemented by conditions at the breaks (the principle of entropy increase) that enable one to choose the unique solution from the class of all discontinuous flows satisfying the equations of motion. We shall describe one of the approaches to the search for this additional principle. This approach is also of interest by itself. It was inspired by Hopf’s paper (1952). Lack of space forces us to omit detailed formulations in this section and to concentrate on the conceptual aspect of the situation as applied to the case of problem (1), (2), (3), (4). Had this problem been uniquely solvable for all \( t \geq 0 \) belonging to a certain functional Banach space \( H \) [\( f(x) \) in this treatment is considered fixed], a nonlinear operator \( \mathbf{V}_t \) transforming points \( \mathbf{a} \) of space \( H \) into points \( \mathbf{v}(x, t) \)—the solutions of problem (1)-(4)—would have been defined for any \( t \geq 0 \). Let \( \mathbf{V}_t \) transform \( H \) into \( H \). Consider a Borelian \( \sigma \)-algebra \( \Sigma \) generated by closed sets. Suppose that transformation \( \mathbf{V}_t \) is measurable with respect to this \( \sigma \)-algebra (for the case of two-dimensional flows all these assumptions are fulfilled). Define on \( (\Sigma, H) \) a probability (normalized) measure \( \mu \) and consider its evolution governed by the equation

\[
\mu_t(\omega) = \mu(\mathbf{V}_{-t}\omega) \tag{23}
\]

for any set \( \omega \) from \( \Sigma \). Here \( \mathbf{V}_{-t}\omega \) is meant to represent the set of all points from \( H \) which are transformed into points of \( \omega \) by \( \mathbf{V}_t \). Thus defined, \( \mu_t \) on \( H \) will also be a probability measure on \((\Sigma, H)\). One can write out a relation connecting \( \mu_t \) with \( \mu \). This may be done in a number of ways. Hopf (1952) has written it out for the characteristic function \( \chi(\theta, t), \theta \in H \) of the measure \( \mu \). It has the form of a linear differential equation for \( \chi(\theta, t) \) with an infinite number of independent variables: \( t \) and \( \theta = (\theta_1, \theta_2, \ldots) \), of the first order in \( t \) and of the second order in variables \( \theta_k \) \( (\theta_1, \theta_2, \ldots \) are special coordinates introduced in \( H \). Foiaş (1973) has written out the connection between \( \mu_t \) and \( \mu \) in the form of an integral identity (the way partial differential equations have been treated for the last two decades). Denote this identity by Roman numeral (I). It turned out that identity (I) is simpler to work with than Hopf’s equation. This identity is linear in the unknown \( \mu_t \). For the case of two-dimensional flows for which operator \( \mathbf{V}_t \) is defined and possesses a number of nice properties including those mentioned above, identity (I) uniquely defines \( \mu_t \) for a fixed \( \mu \) and the relation between them coincides with (23). In other words identity (I) contains the same information on the evolution of the flow as the system (1), (2), (3).

For three-dimensional flows the above reasoning collapses since in that case the (single-valued) operator \( \mathbf{V}_t \) is not defined. Let us base the study of three-dimensional flows not on the Navier-Stokes equations but on identity (I) and concentrate on the evolution of measures \( \mu \) according to (I) rather than on individual solutions to (1), (2), and (3). It turns out that in this approach, to each initial measure \( \mu \) there corresponds a family of probability measures \( \{\mu_t\}, t \in [0, T] \) satisfying identity (I). But this family is possibly not unique. Let \( \{\mu'_t\}, t \in [0, T] \) be another set of probability measures satisfying (I) with the same \( \mu \). Since (I) is linear in \( \mu_t \), it is satisfied (for the same \( \mu \)) by measures \( \gamma \mu_t + (1-\gamma)\mu'_t \), \( t \in [0, T] \) with any \( \gamma \) from the
interval \([0, 1]\), that is, the set of all solutions \(\{\mu_t^x\}, t \in [0, T]\), to identity (I) with fixed \(\mu\) is a convex set in the space of probability measures. We shall denote it by \(\mathcal{M}_t(\mu)\). Let us construct an averaged field of velocities \(\langle \mathbf{v}(t) \rangle = \int_H \mathbf{v}^x(t)(d\nu)\) for each \(\{\mu_t^x\}, t \in [0, T]\), from \(\mathcal{M}_t(\mu)\). At each moment \(t\), points \(\{\langle \mathbf{v}(t) \rangle\}\) form a convex set in \(H\). Choose from \(\{\langle \mathbf{v}(t) \rangle\}\) the \(\langle \mathbf{v}_{\text{min}}(t) \rangle\) that yields the minimal value of the integral

\[
J_T = \int_0^T \int_\Omega \sum_{i,k=1}^3 v_{kx}^2(x, t) \, dx \, dt
\]

(it should be remembered that at each \(t\) the element \(\langle \mathbf{v}(t) \rangle\) belongs to \(H\), i.e., is a vector-function defined for \(x \in \Omega\)). This \(\langle \mathbf{v}_{\text{min}}(t) \rangle\) turns out to exist and to be unique. The “extremum family” of measures \(\{\mu_{\text{min}}^x\}, t \in [0, T]\), corresponding to \(\langle \mathbf{v}_{\text{min}}(t) \rangle\), however, may prove not to be unique. But we are looking for a way to find a unique velocity field, and the field \(\langle \mathbf{v}_{\text{min}}(t) \rangle\), as has just been said, is yielded by the above variational problem uniquely. It satisfies the Reynolds equation in which the averaging is carried out over any “extremum family” of measures \(\{\mu_{\text{min}}^x\}, t \in [0, T]\), corresponding to \(\langle \mathbf{v}_{\text{min}}(t) \rangle\). If the initial measure \(\mu\) had been concentrated in some one point \(a\) of space \(H\), one is inclined to believe that the \(\langle \mathbf{v}_{\text{min}}(t) \rangle\) corresponding to it does in fact describe the evolution of \(a\). If a “not too discontinuous” (and hence unique) solution \(\mathbf{V}_t, a\) to (1), (2), and (3) corresponded over an interval of time \([0, T]\) to \(a\) it would coincide with \(\langle \mathbf{v}_{\text{min}}(t) \rangle\).

The described principle of choosing \(\langle \mathbf{v}_{\text{min}}(t) \rangle\) was formulated by Foiaş and G. Prodi (see Foiaş 1973). It has one essential drawback: it cannot be excluded as yet that the extremum \(\langle \mathbf{v}_{\text{min}}(t) \rangle\) depends on the length of the interval \([0, T]\) on which it is defined. It has not been proved that fields \(\langle \mathbf{v}_{\text{min}}(t) \rangle\), \(t \in [0, T]\), and \(\langle \mathbf{v}_{\text{min}}(t) \rangle\), \(t \in [0, T_1]\), \(T_1 > T\), found according to this principle on intervals \([0, T]\) and \([0, T_1]\) respectively coincide on the smaller interval \([0, T]\). Such a coincidence is essential in our opinion. The situation with regard to this principle is better for solutions \(\{\mu_t^x\}, t \in [0, T]\), that do not depend on \(t\). Such solutions to (I) may naturally be called stationary ones. They are invariant measures for the Navier-Stokes equations. For these solutions \(\langle \mathbf{v}_{\text{min}} \rangle\) is the one field \(\langle \mathbf{v}^x \rangle = \int_H \mathbf{v}^x(d\nu)\) that yields the minimal value of the integral

\[
\int_\Omega \sum_{i,k=1}^3 [v_{kx}^x(x)]^2 \, dx.
\]

Here \(\{\mu_t^x\}\) is a set of all solutions to identity (I) that correspond to the initial measure \(\mu\) and do not depend on \(t\). In this case the above objection to this principle of choice is eliminated.

9 ON SOME DESIRABLE COMPUTER EXPERIMENTS

From Sections 2 and 8 it follows that branching of the solution on \(\mathbf{v}(x, t)\) to problem (1)-(4) may appear only as a result of unlimited increasing of the norm \(\|\mathbf{v}(x, t)\|_{q,\Omega} = [\int_\Omega |\mathbf{v}(x, t)|^q \, dx]^{1/q}\) with some \(q > 3\) (we deal in this paragraph only
with bounded domains $\Omega$ or of the quantity
\[
J(t) = \int_\Omega \sum_{i,k=1}^{3} v_{kx}^2(x, t) \, dx
\]
over a finite interval of time. (We note that for an arbitrary function $u(x)$ the inequality
\[
\|u\|_{q,\Omega} \leq c_q(\Omega) \left[ \int \Omega \left( u^2 + \sum_{i=1}^{3} u_{xi}^2 \right) \, dx \right]^{1/2}
\]
holds if $\Omega$ is a bounded domain in $E_3$ and $q \leq 6$.) In the light of this we think it is desirable to study the behavior of one of these quantities under increasing $t$ for some "good" approximations of the solution to problem (1)-(4). We know of two such approximations. The first is the system of ordinary differential equations obtained when the considered problem is treated by the method of Galerkin (see Ladyzhenskaya 1970, Sect. 4, Chap. VI). Let $v^{(n)}(x, t)$ be the $n$-th Galerkin approximation obtained with some allowed choice of the system of coordinate functions $\{\psi_{k}(x)\}$. It is desirable to make out whether there are such moments of time when $\|v^{(n)}(x, t)\|_{q,\Omega}$ becomes "very large" compared to, for instance, the value
\[
\|v^{(n)}(x, 0)\|_{q,\Omega} \quad \text{or to} \quad \|v^{(n)}(x, t)\|_{q,\Omega} \quad \text{averaged over the time interval [0, T]}
\]
The forces $f$ may be taken as time-independent and smooth with respect to $x$, but the norm $\|f\|_{2,\Omega}$ may not be small, $a(x)$ as smooth and $n$ as "not small." The other approximation is the difference scheme suggested by us for the treatment of general three-dimensional problems. It is described in Ladyzhenskaya (1970, Sect. 9, Chap. VI) (see also Krzywicki & Ladyzhenskaya 1966). [For this scheme it has been proved that its solution $\{v_h\}$ is determined uniquely and that for any way of letting the size of cells of a subdivision of space go to zero one can choose in $\{v_h\}$ a subsequence converging in a certain sense to some (generally speaking discontinuous) solution of problem (1)-(4).] If the solution to the latter is "not too bad" (hence, unique), the complete sequence $\{v_h\}$ will converge to it. For solutions $v_h$ of the difference scheme considered there is a difference analogue of the main energy relation
\[
\int_{\Omega} v^2(x, t) \, dx - \int_{\Omega} v^2(x, t_1) \, dx + 2v \int_{t_1}^{t} \int_{\Omega} \sum_{i,k=1}^{3} v_{kx}^2 \, dx \, dt + 2 \int_{t_1}^{t} \int_{\Omega} f v \, dx \, dt = \int_{\Omega} \int_{t_1}^{t} f v \, dx \, dt,
\]
that holds for any "not too bad" solution to system (1), (2), (3), (4). All this speaks in favor of the suggested difference scheme (later other more economical difference schemes have been suggested, but their convergence was proved only subject to the condition that problem (1)-(4) has a unique smooth solution (see Ladyzhenskaya 1970, Sect. 9, Chap. VI). It is desirable to carry out the following numerical experiment in this scheme on a powerful computer.

The domain $\Omega$ should be taken as a cube with $f(x)$ as some not-too-weak field of forces. The number of cells in the net division should be large. Then $v_h$ corresponding to several initial fields $a(x)$ should be computed keeping track of the value of
If at some moments of time this property becomes "very large" it would indicate that the viscous terms in the Navier-Stokes equations can not prevent the catastrophe. Instead of the norm \( \|v\|_{q,\Omega} \) (or its difference analogue \( \|v_h\|_{q,\Omega_h} \)) one may follow the value of the Dirichlet integral \( J(t) \). It should be stressed that the mentioned norms of the approximations \( v_h(x) \) and \( v_h \) can not become infinite and the critical moments of time are those when these norms become "considerably" larger than, for example, their averages over an interval of time \([0, T]\).

10 SOME GENERALIZATIONS OF THE NAVIER-STOKES EQUATIONS

All the previous sections dealt with the Navier-Stokes equations and did not question the adequacy of the description they provide for the dynamics of real flows. Naturally they are no different from other equations of the theories of continuous media in giving only an approximate description of dynamics. In their derivation it was assumed that the \( |v_{\infty}| \) are "comparatively" small, so we probably should not rely on these equations when these quantities become large. This is true to a greater extent if unbounded velocity gradients may be created in the flow [which in our opinion is not excluded by (1), (2)]. Hence it is natural to wish to substitute the Navier-Stokes system more general equations that would give a better description of flows with large \( |v_{\infty}| \).

Here is an axiomatic approach to the search for such equations. The motion of a continuous medium is described by a system of the following kind:

\[
\rho v_t = \text{div} P + \rho f
\]  

(24)

plus the continuity equation. In (24) \( \rho \) is the density of the medium and \( P = (P_{ij}) \) is the symmetric stress tensor. If Stokes' postulates on the character of dependence of \( P \) on the velocity-deformation tensor \( D = (v_{ij}) \), where \( v_{ij} = v_{ix} + v_{ix} \), is assumed, the following equation (see Serrin 1959) is valid:

\[
P = \alpha(I, I, II, III)E + \beta(I, II, III)D + \gamma(I, II, III)D^2.
\]  

(25)

Here \( \alpha, \beta, \) and \( \gamma \) are scalar functions of the main invariants \( I, II, III \) of tensor \( D \), \( D^2 \) is the square of matrix \( D \), and \( E \) is the unit matrix. In the case of an incompressible liquid

\[
I = \text{div} v = 0, \quad II = \sum_{i,k=1}^3 v_{ik}^2 \equiv \hat{v}^2, \quad III = \det D,
\]

and \( \alpha = -p \) is treated as an unknown function of \( x \) and \( t \) to be determined along with \( v \) from the system of equations describing the motion of the liquid and from the corresponding initial and boundary conditions (the function \( p \) is most often called pressure). Thus for incompressible liquids

\[
P = -pE + \beta(\hat{v}^2, \det D)D + \gamma(\hat{v}^2, \det D)D^2.
\]
In view of this, system (1) for incompressible liquids acquires the form:

\[
\begin{align*}
  v_i + \sum_{k=1}^{3} v_k v_{ik} - \sum_{k=1}^{3} \frac{\partial}{\partial x_k} T_{ik}(v_{ji}) &= -q_{xi} + f_i \\
  \text{div } v &= 0, \quad i = 1, 2, 3,
\end{align*}
\]

where \( T = P + pE \), \( q = p - \frac{1}{2} |v|^2 \), and \( \rho \) is taken to be unity. We have proved (see Ladyzhenskaya 1967b and 1968a) that the system gives a deterministic description of dynamics (that is, initial-boundary-value problems for it of the type (3k), (4) are uniquely solvable in "the large") provided the functions \( T_{ik}(v_{ji}) \) (note that \( T_{ik} = T_{ki} \)) satisfy the following conditions:

1. \( T_{ik}(v_{ji}) \) are continuous functions of \( v_{ji}, j, l = 1, 2, 3, \) and

\[
|T_{ik}(v_{ji})| \leq c(1 + |\vec{v}|^{2\delta}) |\vec{v}|, \quad \delta \geq \frac{4}{3},
\]

where \( |\vec{v}| = \left( \sum_{i,j=1}^{3} v_{ij}^2 \right)^{1/2} \).

2. \( \sum_{i,k=1}^{3} T_{ik}(v_{ji}) v_{ik} \geq v_0 \vec{v}^2 (1 + \varepsilon |\vec{v}|^{2\delta}), \)

where \( v_0 \) and \( \varepsilon \) are positive numbers.

3. For any smooth solenoidal vector-functions \( v'(x) \) and \( v''(x) \) that coincide on the boundary \( \partial \Omega \) the following inequality should be satisfied:

\[
\int_{\Omega} \sum_{i,k=1}^{3} [T_{ik}(v'_{ji}) - T_{ik}(v''_{ji})] (v_{ik} - v'_{ik}) \, dx \geq v_1 \int_{\Omega} \sum_{i,k=1}^{3} (v_{ik} - v'_{ik})^2 \, dx, \quad v_1 = \text{const} > 0.
\]

All these conditions are fulfilled, for example, for functions \( T_{ik}(v_{ji}) = \beta(\vec{v}^2) v_{ik} \), provided the "coefficient of viscosity" \( \beta(\tau) \) is a positive, monotonically increasing function of \( \tau \geq 0 \) for which at large \( \tau \) inequalities \( c_1 \tau^\mu \leq \beta(\tau) \leq c_2 \tau^\mu \) are satisfied with some positive constants \( c_1, c_2 \) and \( \mu \geq \frac{1}{3} \). In particular, \( \beta(\tau) \) may be chosen in the form \( v_0(1 + \varepsilon \tau^2) \), \( v_0, \varepsilon > 0 \). The system that corresponds to it is

\[
v_i + \sum_{k=1}^{3} v_k v_{ik} - v_0 \sum_{k=1}^{3} \frac{\partial}{\partial x_k} [(1 + \varepsilon \vec{v}^2) v_{ik}] = -p_{xi} + f_i
\]

For small \( \varepsilon \vec{v}^2 \) it is close to the Navier-Stokes system. In Ladyzhenskaya (1967b, 1968a) some other arguments connected to the derivation of approximate equations of hydrodynamics from those of Maxwell-Boltzmann are put forward in favor of system (27).

11 HISTORICAL REMARKS

The body of this article does not contain the names of all the scientists who have contributed to mathematical studies of the Navier-Stokes system of equations. Only some of them are mentioned where the presentation calls for it. In this
paragraph it is also impossible to give a complete bibliography of the questions dealt with in the paper because it is too vast. It is even less feasible to try to describe relationships between the works of different authors. We shall mention here only some authors and works of this century who in our opinion have contributed essentially to the material presented in the previous paragraphs; we further limit ourselves only to investigations with mathematically rigorous proofs.

F. K. G. Odqvist in the 1930s created the theory of hydrodynamic potentials for linear stationary problems, which enabled him to reduce the study of the latter to that of Fredholm integral equations of the 2nd kind. His predecessors were Lichtenstein and Oseen. Odqvist has also proved the unique solvability of the nonlinear stationary problems for small Reynolds numbers. The first investigations of the solvability of nonlinear stationary problems for arbitrary Reynolds numbers were carried out in 1933 by J. Leray. Using one of the results of this paper (the a priori estimate of the Dirichlet integral

$$\int \sum_{i,k=1}^{3} \varepsilon_{ik}(x) \, dx$$

for solutions of such problems; a more explicit estimate of this integral was given by E. Hopf in 1941) and the theory of fixed points for completely continuous nonlinear mappings (created by Leray in collaboration with Schauder), Ladyzhenskaya managed to give a comparatively simple proof of the solvability of these problems for any Reynolds numbers (see Ladyzhenskaya 1959 and 1970). Essentially the same result was also proved by I. I. Vorovich & V. I. Yudovich (1961) and H. Fujita (1961).

R. Finn works within the class of classical solutions and following Oseen, Odqvist, and Leray obtains integral representations for solutions through employment and development of the methods of the theory of hydrodynamical potentials. This approach provides more complete information on the behavior of the solution at \( |x| \to \infty \) (see Finn 1965 and Finn & Smith 1967) for small Reynolds numbers.

Important coercive estimates in terms of the norms of various functional spaces for solutions of linearized stationary problems were obtained by Solonnikov (1960, 1961, 1964a, and 1966; see also Ladyzhenskaya 1970). They are useful in the case of nonlinear problems. Part of these estimates is also in the works of Cattabriga (1961) and Vorovich & Yudovich (1961).

The first theorems on the solvability of nonlinear nonstationary problems were established by Leray (1934a). He proved the unique solvability of two-dimensional initial-boundary-value problems for small Reynolds numbers. For large Reynolds numbers he also proved that smooth solutions exist over a small interval of time in the vicinity of smooth initial data. He also indicated some characteristics of the set of those moments when the solution may "become infinite" and start branching.

The author proved in 1958 (Ladyzhenskaya 1958) that there are no such moments, that is, that two-dimensional problems are uniquely solvable for all moments of time and for any Reynolds numbers (see Sect. 2 for details).

Leray (1934b) managed to prove the unique solvability of the three-dimensional Cauchy problem (that is, when the flow fills all infinite space) for either small Reynolds numbers or in the vicinity of smooth initial data. The first result on the
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solvability of general problem (1)-(4) for three-dimensional flows in arbitrary domains belongs to E. Hopf (1951). He proved that this problem always has at least one "weak" solution, which may possess "strong" singularities (in particular its $|v(x, t)|$ may become infinite). Later K. K. Golovkin and Ladyzhenskaya (1960) (see also Ladyzhenskaya 1970 and Solonnikov 1964b) demonstrated that such a solution $v(x, t), p(x, t)$ has generalized derivatives $v_t, v_{xixj}, p_x$, summable over $Q_T = \Omega \times (0, T)$ with the power 5/4. In Sections 2 and 8 we discussed these solutions and said that the uniqueness theorems for this class of "strongly discontinuous" solutions are not proved and in all probability do not hold. The first theorems on unique solvability of the general three-dimensional problem (1)-(4) (see Sect. 2) were proved by Kiselev & Ladyzhenskaya (1957). Further investigations of the smoothness of solutions in relation to that of the data of the problem were carried out by Ladyzhenskaya (1967a, 1970), Solonnikov (1961, 1964a,b), Golovkin (1964) Golovkin & Solonnikov (1961). Prodi (1959, 1962), Lions (1959), Serrin (1959, 1963), Kato & Fujita (1962), Sobolevskii (1964) have analyzed the possibilities of other methods and obtained a number of other characteristics of these solutions including the uniqueness theorems. Nevertheless, all these methods and approaches, in spite of their difference in the study of the main question of unique solvability, are blocked by the same difficulty: the necessity of obtaining an a priori estimate for some one of the norms of solutions $v(x, t)$ (or some approximations to them) indicated in Sect. 8.

The unique solvability of linear nonstationary problems was first proved by the author for some Hilbert spaces (see Ladyzhenskaya 1970). Golovkin & Solonnikov (1961) constructed a theory of three-dimensional nonstationary hydrodynamic potentials (see also Solonnikov 1964b,c). For the two-dimensional case this had been done by J. Leray (1934). Solonnikov (1964b,c) thoroughly investigated the dependence of the solution’s smoothness on the smoothness of the data. He obtained coercive estimates for these solutions in most important functional spaces. These estimates together with the weighted ones (weights of the $e^{-ct}$ type) (see Solonnikov 1973), along with solvability theorems for problems (1)-(4) presented in Sect. 2 have formed the analytical base for the investigation of the stability of the flow dealt with in Sect. 5. The first proof of the correctness of the linearization principle in the question of stability of stationary solutions (see Sect. 5) was given by Prodi (1962). Further results (see Sect. 5) on the general theory of stability for the Navier-Stokes equations were proved by Yudovich (1965b, 1970a,b) and Sattinger (1970). In the paper of Ladyzhenskaya & Solonnikov (1973) these results were obtained for abstract nonlinear equations in Hilbert space and it was proved that a number of hydrodynamic and magneto-hydrodynamic problems for viscous incompressible liquids are special cases of the classes of nonlinear equations they have considered. Sect. 7 is a concise presentation of Ladyzhenskaya (1972). Results discussed in Sect. 8 belong to Prodi and Foiaș and may be found with proofs in papers of Foiaș (1973). The list of references to Sect. 6 is too long to be presented here; some of them are quoted in Sect. 6.

Finally we would like to stress that research carried out by specialists in hydrodynamics proper has been ignored here. Their results derived with mathe-
matematical rigor are well known and may be found in textbooks. Those obtained on the “physical level” of rigor, in spite of the importance of many of them, are outside the scope of this paper.

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