Interior Error Estimates of the Ritz Method for Pseudo-Differential Equations

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The Ritz method for strong elliptic pseudo-differential equations is discussed. 'Optimal' local error estimates are derived if the underlying 'approximation-spaces' are finite elements. The analysis covers simultaneously pseudo-differential operators of positive and negative order. In case of positive order an additional regularity assumption for the 'approximation-spaces' is needed.

Key words: pseudo-differential equations, Ritz method, interior estimates, super-approximability

0. Introduction

Let the linear equation

\[ Au = f \]

be given in a Hilbert-space \( H \) with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). The operator \( A \) is assumed to have the properties

\[ \begin{align*}
  \text{i)} & \quad A \text{ positive, i.e. } (Au, u) > 0 \text{ for } u \neq 0, \\
  \text{ii)} & \quad A \text{ symmetric, i.e. } (Au, v) = (u, Av)
\end{align*} \]

for \( u, v \in D(A) \). Then

\[ a(u, v) := (Au, v) \]

defines an inner product in \( D(A) \). The corresponding norm will be denoted by

\[ \| u \| := a(u, u)^{1/2}. \]

In order to apply the Ritz method we need a closed 'approximation-space'

\[ S = S_n \subseteq H_A := D(A)^{1/2}. \]

The domain of definition of \( a(\cdot, \cdot) \) can be extended to \( H_A \times H_A \). The Ritz approxi-
mation \( u_h := R_h u \in S_h \) is defined by
\[
(0.6) \quad a(u_h, \chi) = (f, \chi) \quad \text{for all } \chi \in S_h.
\]
Because of
\[
(0.7) \quad a(u, v) = (f, v) \quad \text{for all } v \in D(A)
\]
primarily and hence also for \( v \in H_A \) we get the defining relation
\[
(0.8) \quad a(u - u_h, \chi) = 0 \quad \text{for all } \chi \in S_h.
\]
This shows:

The error \( u - u_h \) is orthogonal to \( S_h \) with respect to the inner product \( a(\cdot, \cdot) \). Therefore the Ritz method is best approximating in the norm of \( H_A \), i.e.
\[
(0.9) \quad \| u - u_h \| = \inf_{\chi \in S_h} \| u - \chi \|.
\]
In order to analyze the error \( u - u_h \) further properties of the operator \( A \) resp. the space \( H_A \) are needed.

We will give two illustrations. Regarding the notations we refer to Gilbarg-Trudinger [6].

**Example 1.** \( H = L_2(\Omega) \) with \( \Omega \subseteq \mathbb{R}^n \) a bounded domain and the boundary \( \partial \Omega \) sufficiently smooth.
\[
(0.10) \quad Au := -\Delta u \quad \text{and} \quad D(A) = \dot{W}^2_2(\Omega) := W^2_2(\Omega) \cap W^1_2(\Omega).
\]
We introduce a Hilbert-scale in the following way: Let \( \{ \psi_i, \lambda_i \} \) be the orthonormal set of eigen-pairs of \( A \), i.e.
\[
(0.11) \quad \begin{align*}
-\Delta \psi_i &= \lambda_i \psi_i & \text{in } \Omega \\
\psi_i &= 0 & \text{on } \partial \Omega.
\end{align*}
\]
The Hilbert-spaces \( \{ H_\beta \mid \beta \in \mathbb{R} \} \) are spanned by the functions with a finite \( \beta \)-norm defined by
\[
(0.12) \quad \| z \|_\beta^2 := \sum \lambda_i^{\beta} \| z_i \|^2 \quad \text{with} \quad z_i := (z, \psi_i).
\]
We have the inclusions
\[
(0.13) \quad D(A) \subseteq H_A = H_0 = \dot{W}^2_2(\Omega) \subseteq L_2(\Omega).
\]

**Example 2.** \( H = L_2(\Gamma) \) with \( \Gamma = S^1(\mathbb{R}^2) \), i.e. \( \Gamma \) is the boundary of the unit sphere. Then \( H \) is the space of \( L_2 \)-integrable periodic functions in \( \mathbb{R} \).
\[
(0.14) \quad Au(x) = \int k(x - y)u(y)dy \quad \text{and} \quad D(A) = H
\]
with
(0.15) \[ k(y) := -\ln \left| 2 \sin \frac{y}{2} \right|. \]

With the help of the Fourier coefficients \( v_\alpha \) of a \( 2\pi \)-periodic function \( v \) defined by

\[ v_\alpha := \frac{1}{2\pi} \int_{-\pi}^{\pi} v(x)e^{-i\alpha x}dx \]

we may introduce for real \( \beta \) the norms

\[ \|v\|^2_\beta := \sum_{\alpha=-\infty}^{\infty} |v|^{2\beta} |v_\alpha|^2. \]

The Hilbert-spaces \( H_\beta = H_\beta(\Gamma) \) are defined similar to the above. The Fourier coefficients of the convolution \( Au \) (0.14) are

\[ (Au)_\alpha = k_\alpha \cdot u_\alpha = \frac{1}{2|v|} \cdot u_\alpha. \]

This time we have (see Hsiao-Wendland [8])

\[ D(A) \subseteq H_\alpha = H_{-1,2}(\Gamma). \]

The meaning of local convergence will be demonstrated in case of Example 1. We use isoparametric finite elements

\[ S_h \subseteq W^1_2(\Omega) \]

which are piecewise polynomials of degree less than \( t \) (see Zlamal [20]). The index \( h \in (0, 1] \) is a measure of the underlying subdivision. In Nitsche-Schatz [14] it is shown:

Let \( u \in W^1_2(\Omega) \) be the solution of (0.10) with the additional regularity \( u \in W^1_2(\Omega_2) \) for \( \tau = t \) and some subdomain \( \Omega_2 \subseteq \Omega \). In a proper subdomain \( \Omega_1 \subseteq \Omega_2 \), the error estimate

\[ \|u-u_h\|_{W^1_2(\Omega)} \leq ch^{-\kappa} \left( \|u\|_{W^1_2(\Omega)} + |u|_{W^1_2(\Omega)} \right) \]

for \( \kappa = 0, 1 \) holds true.

In order to get this 'optimal' local convergence a special super-approximability property of finite elements is used. In addition certain global shift properties of the operator \( -\Delta \) are needed.

The construction of an operator-algebra consisting of integral and differential operators leads to the concept of pseudo-differential operators. The counterpart of (0.2) resp. (0.4) regarding the application of the Ritz method is Gårding's inequality for strong elliptic pseudo-differential operators (see Schatz [16]). For such equations the Ritz method is almost best approximating with respect to the corresponding 'energy-norm'. The Examples 1 and 2 are model problems with strong elliptic pseudo-differential operators of order \( 2\alpha \) and the 'energy-norm' \( \| \cdot \|_\alpha \) with \( \alpha = 1 \) resp. \( \alpha = -1/2 \).
In the present paper we will derive the 'optimal' local convergence of the Ritz method for strong elliptic pseudo-differential operators. We emphasize that our treatment covers simultaneously operators of positive and negative order.

We will use the local shift properties of elliptic pseudo-differential operators $P$ (see Treves [18] p. 42):

Let $w$ be the solution of an elliptic pseudo-differential equation

$Pw = f$. If $f$ is in $C^2(\Omega')$ for some open domain $\Omega'$ then also

$w \in C^2(\Omega')$.

1. Global Error Estimates

Let $\{H_\beta\ : \ beta \in \mathbb{R}\}$ be a Hilbert-scale with the special assumption:

For $\beta = m \in \mathbb{N}_0$ (the set of all nonnegative integers) the spaces

\[
H_m \equiv W^m_2(\Omega)
\]

are subspaces of the Sobolev-space $W^m_2(\Omega)$ with

\[
\Omega = \Sigma \quad \text{or} \quad \Omega = \partial \Sigma
\]

and $\Sigma$ being a bounded domain with boundary $\partial \Sigma$ sufficiently smooth.

$(\cdot, \cdot)_\beta, \| \cdot \|_\beta$ will denote the inner product respectively the norm in $H_\beta$. In case of $\beta = 0$ we skip the subscript.

We assume that the operator $A$ (0.1) has the following properties:

1) There is an $\alpha \in \mathbb{R}$ such that

i) The mapping $A: H_{\beta+2\alpha} \rightarrow H_\beta$ is an isomorphism for $\beta \in \mathbb{R}$, i.e.

\[
c^{-1} \| u \|_{\alpha+2\alpha} \leq \| Au \|_\beta \leq c \| u \|_{\beta+2\alpha}
\]

with some constant $c$.

ii) $A$ is positive definite in $H_\alpha$, i.e.

\[
(u, Au) \geq \epsilon \| u \|_\alpha^2
\]

with $\epsilon > 0$.

2) $A$ is self-adjoint in $H = H_0$, i.e.

\[
(Au, v) = (u, Av)
\]

for $u, v \in D(A)$.

By

\[
a(u, v) = (Au, v) \quad \text{for} \quad u, v \in D(A)
\]

an inner product is defined.

Lemma 1.1. There is a constant $c$ (depending on $\beta$) such that
\[(1.5) \quad c^{-1}\|u\|_\beta \leq \sup_{v \in H_\beta, \|v\|_\beta > 0} \frac{a(u, v)}{\|v\|_\beta} \leq c\|u\|_\beta \quad \text{for} \quad u \in H_\beta \]

with \( \beta^* := 2\alpha - \beta \).

Remark 1.2. In the following we will denote with \( c \) numerical constants which may differ at different places.

Proof. The right part of the inequality (1.5) is a direct consequence of Schwarz’ inequality and (1.3i).

By the standard inequality in Hilbert-scales we have

\[(1.6) \quad \|u\|_\beta \leq c \sup_{w \in H_\beta, \|w\|_\beta > 0} \frac{(u, w)}{\|w\|_\beta} .\]

In order to show the left inequality we define for \( w \in H_\beta \) an auxiliary function \( v \) by \( Av = w \). On the one hand it is \((u, w) = a(u, v)\) and on the other hand

\[(1.7) \quad \|v\|_\beta \leq \|Av\|_\beta \leq c\|w\|_\beta .\]

Since we will consider only ‘approximation-spaces’ which are contained in \( L_2 = H \) we impose the following regularity in order to apply the Ritz method:

\[(1.8) \quad S_h \subseteq H_\beta \quad \text{with} \quad \alpha := \max\{0, \alpha\} .\]

In our analysis we will need the regularity

\[(1.9) \quad S_h \subseteq H_\beta \quad \text{with} \quad s := \begin{cases} 2\alpha, & 2\alpha \in N_0 \\ [2\alpha] + 1, & 2\alpha \notin N_0 \end{cases} \]

which is an additional assumption only in case of \( \alpha > 0 \). For any linear bounded operator \( B : H_\beta \rightarrow H_\beta \) we introduce the norm

\[(1.10) \quad \|B\|_\beta := \sup_{u \neq 0} \frac{\|Bu\|_\beta}{\|u\|_\beta} .\]

Theorem 1.3. The Ritz operator \( R_h : H_\beta \rightarrow H_\beta \) defined by (0.6) admits for \( \beta, \gamma \in [s^*, s] \) with \( s^* := 2\alpha - s \) the estimate

\[(1.11) \quad c^{-1}\|R_h\|_{\beta, \gamma} \leq \|R_h\|_{\gamma, \beta} \leq c\|R_h\|_{\beta, \gamma} .\]

Proof. Because of \((\beta^*)^* = \beta\) and \((\gamma^*)^* = \gamma\) it is sufficient to show one of the inequalities. Using (0.8) and (1.5) we get

\[(1.12) \quad \|R_h\|_{\beta, \gamma} \leq \sup_{v \in H_\gamma, \|v\|_\gamma > 0} \frac{\|R_h v\|_\beta}{\|v\|_\gamma} \leq c \sup_{v \in H_\gamma, \|v\|_\gamma > 0} \frac{\|R_h v\|_\beta}{\|v\|_\gamma} \leq c \sup_{v \in H_\beta, \|v\|_\beta > 0} \frac{\|R_h v\|_\gamma}{\|v\|_\beta} = c\|R_h\|_{\beta, \gamma} .\]
By
\begin{equation}
N_\beta(\psi) := \sup_{x \in S_h} \frac{a(\psi, x)}{\| x \|_\beta^\nu} \quad \text{for} \quad \psi \in S_h
\end{equation}
a norm is defined in \( S_h \). For \( S_h \) finite dimensional this new norm is equivalent to the \( \beta \)-norm. Obviously we have
\begin{equation}
N_\beta(\psi) \leq c \| \psi \|_\beta .
\end{equation}
We introduce \( \kappa_h \) by
\begin{equation}
\kappa_h := \sup \{ \| \psi \|_\beta \mid \psi \in S_h, \quad N_\beta(\psi) = 1 \}
\end{equation}
and show
\begin{equation}
\text{THEOREM 1.4.} \quad \text{The following assertions are equivalent:}
\end{equation}
\begin{enumerate}
\item \( \| R_h \|_{\beta \rightarrow \beta} \leq c \),
\item \( \kappa_h := \kappa_h^{-1} \geq \tau > 0 \) (with \( \tau \) independent of \( h \),
\item \( \inf \sup_{\psi \in S_h, \quad x \in S_h} \frac{a(\psi, x)}{\| x \|_\beta, \| \psi \|_\beta} \geq \tau > 0 \quad \text{for each} \ \beta \in [\sigma, s] \).
\end{enumerate}
\begin{rem}
Remark 1.5. Theorem 1.4 may be considered as a generalization of the Polskii condition (see Polskii [15]). We notice that (1.16i) holds if and only if the Ritz method is almost best approximating in the \( \beta \)-norm (see Alexits [1]). With respect to (1.16ii-iii) we refer to Aziz-Kellog [3].
\end{rem}
\begin{proof}
(0.8) and Theorem 1.3 give for \( \psi \in S_h \)
\begin{equation}
\| \psi \|_\beta \leq c \sup_{v \in H_0^\sigma} \frac{a(\psi, v)}{\| v \|_\beta} = c \sup_{v \in H_0^\sigma} \frac{a(\psi, R_h v)}{\| R_h v \|_\beta}
\end{equation}
\begin{equation}
\leq c N_\beta(\psi) \| R_h \|_{\beta \rightarrow \beta},
\end{equation}
which shows
\begin{equation}
\kappa_h \leq c \| R_h \|_{\beta \rightarrow \beta} \leq c \| R_h \|_{\beta \rightarrow \beta} .
\end{equation}
On the other hand we find with (0.8) for \( u_h = R_h u \in S_h \)
\begin{equation}
\| u_h \|_\beta \leq \kappa_h N_\beta(u_h)
\end{equation}
\begin{equation}
\ni \sup_{x \in S_h} \frac{a(u, x)}{\| x \|_\beta^\nu} \leq c \kappa_h \| u \|_\beta
\end{equation}
and therefore
Thus the equivalence of (1.16(i)) and (1.16(ii)) is shown. The equivalence of (1.16(ii)) and (1.16(iii)) is obvious.

We will use certain approximation properties of the spaces $S_h$.

**DEFINITION 1.6.** We use the notation $S_h = S_h^{k+1}$ with $k < t$ if the following statements hold true:

i) $S_h = H_h$.

(1.21) ii) $\inf_{\mathcal{S}_h} \|v - \chi\|_h \leq c h^{t-k} \|v\|_t$ for $v \in H_t$.

iii) $\|\chi\|_h \leq c h^{-k} \|\chi\|_t$ for $\chi \in S_h$ for $k' < k$.

**REMARK 1.7.** In the one dimensional case the trigonometric polynomials of degree $n$ share these properties with $h = n^{-1}$ for any $(k, t)$.

**REMARK 1.8.** If $S_h$ is spanned by piecewise polynomial functions subject to regular subdivision of $\Omega$ then the conditions of Definition 1.6 hold true if the elements of $S_h$ are global in $C^{k-1}$ and the degree of the polynomials is at least $t-1$.

The Bramble-Scott-Lemma (see [5]) gives

**LEMMA 1.9.** Let $\beta, \beta'$ be fixed with $\beta < \beta' \leq k$. To any $v \in H_t$ with $\beta \leq \tau \leq t$ there exists a $\chi \in S_h^{k-1}$ with

$$
(v - \chi)_{\beta} \leq c h^{t-\beta} \|v\|_t
$$

simultaneously for $\beta \in [\beta, \beta']$.

With the help of the logarithmic convexity of the norms in a Hilbert-scale the inverse properties

(1.23) $\|\chi\|_{\beta} \leq c h^{-\beta} \|\chi\|_{\beta'}$ for $\chi \in S_h$

are valid for any pair $(\beta', \beta)$ with $\beta' \leq \beta \leq k$ and $\beta' \leq \beta'$. The standard error estimates in our setting are —see the assumptions (1.3)—

**THEOREM 1.10.** Let $u_h \in S_h^{k+1} \subseteq H_h$ be the Ritz approximation on a function $u \in H_t$ with $\beta \leq \tau \leq t$. Then the error estimate

(1.24) $\|u - u_h\|_\beta \leq c h^{t-\beta} \|u\|_t$

holds for $\beta \in [\alpha, k]$ with $t^* = 2\alpha - t$.

**REMARK 1.11.** Up to now our only assumptions on $S_h$ are the approximation properties of Definition 1.6. As a consequence the error estimate (1.24) is valid for conforming finite element methods, boundary finite element methods, spectral and
pseudospectral finite element methods.

2. Local Error Estimates

In addition to (1.3) we assume that the following properties are valid:

i) Let $\Omega' \subseteq \Omega$ and $Au \in C^{\infty}(\Omega') = \bigcap_{\rho \in \mathbb{R}} H_{\rho}(\Omega')$, then $v \in C^{\infty}(\Omega')$.

ii) Let $\rho$ and $\sigma$ be cut-off functions, i.e. $\rho, \sigma \in C_0^{\infty}(\Omega)$, with $\text{supp}(\rho) \cap \text{supp}(\sigma) = \emptyset$. To any pair $\beta, m \in \mathbb{R}$ there is a constant $c$ with

$$|\rho A v|_{\beta + m} \leq c \|v\|_\beta \quad \text{for } v \in H_\beta.$$  

iii) The operator $A_2 := \omega A - A + \omega$ with $\omega \in C^{\infty}(\Omega)$ is of order $2z - 1$, i.e.

$$|A_2 v|_{\beta} \leq c \|v\|_{\beta + 2z - 1}$$

for any $\beta \in \mathbb{R}$.

The following local approximation properties of the spaces $S_h = S_h^{k+1}(\Omega)$ are typical for finite elements with $\kappa$-regular subdivision (see Nitsche-Schatz [13] and the literature cited):

E.1 [Local Approximability]. Let $v \in H_\tau(\Omega)$ with $\tau \leq t$ be fixed and let $\Omega_1 := \text{supp}(v) \subseteq \Omega$ be contained properly in $\Omega$. There exists a second domain $\Omega' \subseteq \Omega$ and $h_0 > 0$ depending on $\text{dist}(\Omega_1, \Omega')$ such that for $h \leq h_0$ there exists a $\chi \in S_h$ with

i) $\text{supp}(\chi) \subseteq \Omega'$,

ii) $\text{dist}(\Omega_1, \text{supp}(\chi)) \leq c h$,

iii) $\|v - \chi\|_{L^1(\Omega')} \leq c h^{\kappa - 1/2} \|v\|_{L^1(\Omega)}$.

for integer $\lambda$ with $0 \leq \lambda \leq k$ and $\lambda < \tau$.

The constant $c$ is independent of $v$ and $h$.

E.2 [Super-Approximability]. Let $\omega \in C^{\infty}(\Omega)$ be fixed such that the inclusions $\Omega_1 := \text{supp}(\omega) \subseteq \Omega$ hold. There is an $h_0 > 0$ (depending on $\text{dist}(\Omega_1, \Omega')$) such that for $h \leq h_0$ the function $\omega \varphi$ with $\varphi \in S_h$ arbitrary can be approximated by a function $\chi \in S_h$ such that

\begin{align*}
(2.3) & \quad \text{i) } \text{supp}(\chi) \subseteq \Omega', \\
& \quad \text{ii) } \|\omega \varphi - \chi\|_{L^1(\Omega')} \leq c h^{\lambda + 1/2} \|\varphi\|_{L^1(\Omega')} \quad \text{for } 0 \leq \lambda \leq k.
\end{align*}

The constant $c$ will depend only on $\omega$ and its derivatives up to order $k$ as well as on the distance $\text{dist}(\Omega_1, \Omega')$.

A consequence of E.1 is (see Nitsche-Schatz [13]):

LEMMA 2.1. Let $v \in H_\tau(\Omega_2)$ with $\tau \leq t$ and $\Omega_1 \subseteq \Omega_2 \subseteq \Omega$ be given. There exists a $\chi \in S_h$ such that
\[(2.4)\] 
\[i) \ \text{supp}(\chi) \subseteq \Omega_2, \]
\[ii) \ \|v - \chi\|_{\lambda; \Omega_2} \leq ch^{k-\lambda}\|v\|_{\lambda; \Omega_2} \quad \text{for } 0 \leq \lambda \leq k \text{ and } \lambda < \tau.\]

The super-approximability (2.3) is restricted to integer \(k\). For ‘negative’ norms we will show

**Lemma 2.2.** Let \(\omega, \Omega_1, \Omega_2, \text{ and } h_0\) be as in E.2. Further let \(P_h\) be the orthogonal projection of \(H\) onto \(S_h\). Then for \(\varphi \in S_h\) arbitrary the estimate
\[(2.5)\]
\[\|\omega \varphi - P_h(\omega \varphi)\|_{-l} \leq c h^l \|\varphi\|_{-l},\]
holds true for real \(l\) with \(0 \leq l \leq t\) and \(c\) independent of \(\varphi\) and \(h\).

**Proof.** Because of the characterization
\[(2.6)\]
\[\|\omega \varphi - P_h(\omega \varphi)\|_{-l} = \sup_{v \in H_l, v \neq 0} \frac{\langle \omega \varphi - P_h(\omega \varphi), v \rangle}{\|v\|_l},\]
and the fact that \(P_h\) is the orthogonal projection we get with \(\chi \in S_h\) arbitrary
\[(2.7)\]
\[\|\omega \varphi - P_h(\omega \varphi)\|_{-l} = \sup_{v \in H_l, v \neq 0} \frac{\langle \omega \varphi - P_h(\omega \varphi), v - \chi \rangle}{\|v\|_l}.\]

We choose \(\chi \in S_h\) corresponding to the approximation properties (1.21\(\beta\)) of the spaces \(S_h\) and get with (2.3) and (1.23)
\[(2.8)\]
\[\|\omega \varphi - P_h(\omega \varphi)\|_{-l} \leq c h^l \|\omega \varphi - P_h(\omega \varphi)\|_0\]
\[\leq c h^{l+1} \|\varphi\|_0\]
\[\leq c h^l \|\varphi\|_{-l}\]

In the proof of Lemma 2.4 below we will apply Lemma 2.2 in case of \(z < 0\) with \(l = 2|z|\). In case of \(z > 0\) we consider for simplicity only \(z\) with \(z \in N\). In order to do this the superscripts \(k\) and \(t\) characterizing the spaces \(S_k = S_k^t(\Omega)\) are subject to
\[(2.9)\]
\[0 \leq k = 2z < t \quad \text{for } z > 0\]
\[0 = k < 2|z| \leq t \quad \text{for } z < 0.\]

The main result of our paper is

**Theorem 2.3.** Let \(u\) be the solution of (0.1) and assume the regularity \(u \in H^a(\Omega) \cap H^t(\Omega_2)\) with \(a < t\) and \(\Omega_2 \subseteq \Omega\). Further let \(\Omega_1\) be a second domain with \(\Omega_1 \subseteq \Omega_2\) and \(h_0\) chosen properly. The error \(E = u - u_h\) between \(u\) and the Ritz approximation \(u_h\) (0.6) admits for \(h \leq h_0\) the local estimate \((t^* = 2z - t)\)
\[(2.10)\]
\[\|E\|_{0, \Omega_1} \leq c \left[ h^l \|u\|_{0, \Omega_2} + \|u\|_{a, \Omega_2} + \|E\|_t + h^{t-a} \inf_{\varphi \in S_h} \|u - \varphi\|_t \right].\]

In proving the theorem the essential step is
Lemma 2.4. Let \( u, \tau, \Omega \) etc. be as in Theorem 2.3 and let \( \Omega'_2 \) be chosen such that \( \Omega'_2 \in \Omega \subset \Omega'_2 \). Then

\[
\|E\|_{0, \Omega_2} \leq c \left( \|h\|_{L^\infty} + \inf_{x \in S_n} \|u - \chi_x\|_{a, 0} \right) + \frac{c}{h'} \|E\|_{0, \Omega_2}. 
\]

Before proving the lemma we show that Theorem 2.3 is a consequence:

Let additional domains be chosen such that

\[
\Omega'_1 \in \Omega'_2 : \Omega'_1 \in \cdots \in \Omega'_i : \cdots = \Omega_2,
\]

then we apply Lemma 2.4 successively with \( \Omega'_1, \Omega'_2 \) replaced by \( \Omega'_i, \Omega'_i, \cdots \), which finally gives the inequality stated in Theorem 2.3 (since \( \|E\|_{0, \Omega_2} \leq \|E\|_{a, 0} \leq c \|u\|_{a, 0} \leq c \|u\|_{a, 0} \)).

In order to prove Lemma 2.4 we consider similarly additional domains \( \Omega'_i \) as above in the following way

\[
\Omega'_1 : \Omega'_1 \in \cdots \in \Omega'_i : \cdots = \Omega_2.
\]

Let \( \omega_i \in C^0(\Omega) \) (1 \( \leq i \leq 8 \)) be cut-off functions with respect to \( \Omega'_i \) and \( \Omega'_i, \cdots \) such that

i) \( \omega_i \equiv 1 \) in \( \Omega'_i \),
ii) \( \text{supp}(\omega_i) \in \Omega'_i, \cdots \),
iii) \( 0 \leq \omega_i \leq 1 \),

and put

\[
\omega_i := 1 - \omega_i.
\]

With the help of an appropriate approximation \( \Psi \in S_\infty \) on \( u \) we use the splitting

\[
E = (u - \Psi) - (u_0 - \Psi)
\]

\[
\theta = \phi - \Psi.
\]

Because of Lemma 2.1 we may choose \( \Psi \) such that

\[
\|\theta\|_{0, \Omega_2} \leq c\|h\|_{L^\infty} \|u\|_{a, \Omega_2}.
\]

With the help of \( \omega_i \) we may estimate

\[
\|E\|_{0, \Omega_2} \leq \|E\|_{a, \Omega_2} = (\omega_i, E, E).
\]

Let the auxiliary function \( w \) be defined by

\[
Aw = \omega_i, E \in H.
\]

Because of (1.3) and (2.1) we have the regularity

\[
w \in H_{2, \infty}(\Omega) \cap C^\infty(\Omega - \Omega'_2).
\]

We denote by \( w_h := R_h w \in S_h \) the Ritz approximation on \( w \). For the error

\[
e := w - w_h
\]
we have the defining relation
\[(A\varepsilon, \chi) = 0 \quad \text{for all} \quad \chi \in S_h.\]

Analogue to (2.16) we use the splitting
\[
e = (w - \psi) - (w - \tilde{\psi}) \quad \text{with} \quad \psi \in S_h
\]
\[
= : \varepsilon - \varphi.
\]
The choice of \(\psi\) is crucial. According to the representation \(w = \omega_2 w + \omega_2 w\) we use \(\psi = \psi_1 + \tilde{\psi}_2 \in S_h\) with \(\psi_2\), \(\tilde{\psi}_2\) defined by
\[
(\text{i}) \quad \psi_2 \in S_h \text{ is an approximation on } \omega_2 w \text{ with } \text{supp}(\psi_2) \subseteq \Omega'_4 \text{ according to the local approximability E.1,}
\]
\[
(\text{ii}) \quad \tilde{\psi}_2 \in S_h \text{ is an approximation on } \tilde{\omega}_2 w \in C^m(\Omega) \text{ according to Lemma 2.1.}
\]
For—see (2.23)—
\[
e = (w_2 w - \psi_2) + (\tilde{\omega}_2 w - \tilde{\psi}_2)
\]
\[
= : \varepsilon + \tilde{\varepsilon}_2
\]
we get—see (2.19), (2.20)—
\[
\text{i) supp}(\varepsilon) \subseteq \Omega'_4, \quad \|\varepsilon\|_{2,2} \leq c \|E\|_{\omega_1},
\]
\[
\text{ii) } \|\tilde{\varepsilon}_2\|_a \leq ch^{-a} \|E\|_{\omega_1}.
\]
Now we turn to the

**Proof of Lemma 2.4.** Because of (6.8) we get from (2.19) with any \(\chi \in S_h\)
\[
\|E\|_{2,2}^2 \leq \|E\|_{\omega_1}^2 = (A\varepsilon, w) = (A\varepsilon, w - \chi).
\]
The special choice \(\chi = \psi \in S_h\)—see (2.23)—leads to
\[
\|E\|_{2,2}^2 = (E, A\varepsilon)
\]
which we split as follows
\[
\|E\|_{2,2}^2 = (E, \tilde{\omega}_3 A\varepsilon) + (E, \omega_5 A\varepsilon)
\]
\[
= : T_1 + T_2.
\]
Using Theorem 1.10, (2.1ii), (2.26), and the fact that \(\omega_5\) and \(\omega_5 + 1\) have disjoined supports we come to the following sequence of inequalities for the first term \(T_1\) on the right hand side in (2.29)
\[
|T_1| = |(E, \tilde{\omega}_3 A\omega_5 A\varepsilon) + (E, \omega_5 A\tilde{\omega}_4 A\varepsilon)|
\]
\[
\leq c \|E\|_a \|\tilde{\omega}_3 A\omega_5 A\varepsilon\|_a + \|\omega_5 A\tilde{\omega}_4 A\varepsilon\|_{\omega_5 A\tilde{\omega}_4 A\varepsilon}\]
\[
\leq c \|E\|_a \|\varepsilon\|_{2,2} + \|\omega_5 A\tilde{\omega}_4 A\varepsilon\|_{\omega_5 A\tilde{\omega}_4 A\varepsilon}.
\]
\[
\begin{align*}
&\leq c \left\{ \|E\|_r \|w\|_{2s} + \|E\|_0 \|\dot{\varphi}\|_a \right\} \\
&\leq c \|E\|_{a_1} \left\{ \|E\|_r + h^{s-\epsilon} \inf_{u \in \mathcal{S}_h} \|u - \varchi\|_a \right\}.
\end{align*}
\]

In order to estimate the second term \(T_2\) we use the identity

\[
(2.31) \quad T_2 = (\omega_2 E, A\varphi)
\]

\[
= (\omega_2 E, A\varphi) + (E, \omega_2 A\varphi)
\]

\[
= (\omega_2 \theta, A\varphi) - (\omega_2 \phi, A\varphi) + (E, \omega_2 A - A\omega_2)\varphi + (E, A\varphi).
\]

Because of the defining relations for \(E\) and \(e\) we can rewrite \(T_2\) with \(\xi, \eta \in S_h\) arbitrary

\[
(2.32) \quad T_2 = (\omega_2 \theta, A\varphi) - (\omega_2 \phi - \varxi, A\varphi) + (E, A\varphi) + (AE, \omega_2 \phi - \eta).
\]

Here \(A_2\) is defined by \(A_2 := \omega_2 A - A\omega_2\). We choose \(\xi, \eta \in S_h\) such that the superapproximability property (2.3) with \(\omega_2\), \(\varphi\) replaced by \(\omega_2\), \(\Phi\) is fulfilled. With the help of (2.16), (2.17), and Theorem 1.10 we find the bound needed for the first two terms in (2.32)

\[
(2.33) \quad |(\omega_2 \theta, A\varphi) - (\omega_2 \phi - \varxi, A\varphi)| \leq |AE|_0 \|\theta\|_{a_2} + \|\omega_2 \phi - \varxi\|_0 \leq c \|E\|_{a_1} \{h^{s} \|u\|_{a_1} + h |E|_0 \|\omega_2\|_{a_2}\}.
\]

The third term in (2.32) can be estimated with the help of (2.1ii) and (2.1iii)

\[
|E, A_2\varphi| = |(E, \omega_0 A_2\varphi) + (E, \omega_0 A_2\varphi)|
\]

\[
= |(E, \omega_0 A_2\varphi) - (E, \omega_0 A_2\varphi)|
\]

\[
\leq \|E\|_{a_0} \|A_2\varphi\|_0 + \|E\|_r \|\omega_0 A_2\varphi\|_r.
\]

\[
(2.34) \quad \leq c \left\{ \|E\|_{a_0} \|\varphi\|_{2a_1} + \|E\|_r \|\varphi\|_{2a_1} \right\}
\]

\[
\leq c \left\{ \|w\|_{2s} (h \|E\|_{a_0} + |E|_r) \right\}
\]

\[
\leq c \|E\|_{a_1} \{h \|E\|_{a_0, a_2} + |E|_r\}.
\]

In order to estimate the fourth term in (2.32) we choose

\[
(2.35) \quad \eta := P_h(\omega_2 \varphi).
\]

Since the \(L_2\)-projection has 'optimal' local convergence (see Nitsche-Schatz [13]) we get

\[
(2.36) \quad \|\omega_2 (\omega_2 \varphi - \eta)\|_a \leq c h^{s-\epsilon} \|E\|_{a_1}.
\]

Using (1.3i), (2.3) resp. (2.5), (2.36), and Theorem 1.10 we come to the final sequence of inequalities
\[(AE, \omega_2 \varphi - \eta)\]
\[
=|\langle E, A(\omega_2 \varphi - \eta)\rangle|
\leq \|E\|_{\omega_2} \|A(\omega_2 \varphi - \eta)\|_0 + \|E\|_\omega \|\omega_2 A\varphi - \eta\|_\omega + \|E\|_\omega \|A\eta\|_\omega
\leq c\left(\|E\|_{\omega_2} \|\omega_2 \varphi - \eta\|_{\omega_2} + \|E\|_\omega \|\omega_2 \varphi - \eta\|_{\omega_2} + \|E\|_\omega \|\omega_2 \eta\|_\omega\right)
\leq c\left(h|E|_{\omega_2} + \|E\|_\omega \|\omega_2 \varphi - \eta\|_{\omega_2} + \|E\|_\omega \|\omega_2 \eta\|_\omega\right).
\]

This completes the proof of Lemma 2.4.

We will use Example 2 to give an illustration of Theorem 2.3: Because of $x = -\frac{1}{2}$ and therefore $a = \max(0, \sigma) = 0$ condition (2.9) for the superscripts $k$ and $t$ characterizing the spaces $S_k^{a, t}$ leads to

\[(2.38)\]

\[0 = k < t.\]

For a (uniform) $\kappa$-regular subdivision $\gamma = \gamma_n$ of the interval $I = (-\pi, \pi)$ we consider piecewise linear, periodic splines

\[(2.39)\]

\[S_2 := S_n^{0, 2}(I) \subset L^2(I).\]

Let $u \in H^2(I) \cap H(I,I)$ with $I \subset I$ and $0 \leq \sigma < \tau \leq 2$ be the periodic solution of (0.14). Then the third and fourth term on the right hand side in (2.10) give (if $t^* = 2z - t = -3$)

\[(2.40)\]

\[h^{t^* - a} \inf_{\kappa \in \kappa_n} \|u - \chi\|_\kappa \leq h^2 \|u\|_0.\]

Therefore we only need the global regularity assumption $(\sigma = 0) u \in L^2(I)$ in order to get the 'optimal' local error estimate

\[(2.41)\]

\[\|u - u_n\|_{0, I} \leq ch^2 \{\|u\|_{1, I} + \|u\|_{0, I}\}
\]

for $I \subseteq I$. 

References


