LINEAR PARABOLIC EQUATIONS WITH A SINGULAR LOWER ORDER COEFFICIENT

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ABSTRACT

Let \( a, b \) belong to the Hölder class \( H^{\alpha/2}([0,1] \times [0,T]) \) with \( \alpha \in (0,1) \) and \( a > a_0 > 0 \). It is shown that for the solution of the problem

\[
  \begin{aligned}
    u_t - au_{xx} - b u_x = t^{-1/2} f, & \quad (x,t) \in [0,1] \times [0,T], \\
    u(\cdot,0) = \phi, & \\
    u(\cdot,t) = \psi_v, & \quad v = 0,1,
  \end{aligned}
\]

the estimate

\[
  \|u, u_t, t^{1/2}u_{xx}, \alpha, \alpha/2 \| <
\]

\[
  c(\|f\|_{H^{\alpha/2}}^2, \|\phi\|_{H^{\alpha}} + \sum_{v=0}^1 \|\psi_v\|_{H^{\alpha/2}}^2)
\]

holds if the data \( f, \phi, \psi_v \) satisfy the appropriate compatibility condition.

Here \( \| \cdot \|_{H^{\alpha, \alpha/2}} \) denote the norms of the Hölder classes \( H^{\alpha, \alpha/2}([0,1] \times [0,T]), H^{\alpha}([0,1]) \) and \( H^{\alpha/2}([0,T]) \) respectively and the constant \( c \) depends on \( \alpha, a_0, \|a, b\|_{H^{\alpha, \alpha/2}} \) and \( T \). The result extends to quasilinear problems with \( a, b \) and \( f \) depending on \( x, t \) and \( u \).

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Work Unit Number 1 - Applied Analysis
This report was motivated by the study of free boundary value problems related to the Stefan problem. The precise description of the smoothness of the free boundary requires sharp regularity results for linear parabolic equations with singular coefficients.

These results are of independent interest. They can be applied to parabolic equations on domains with curved boundaries that touch the x-axis. As a particular example consider the heat equation

$$u_t - u_{xx} = t^{-1/2} f \text{ on } \Omega$$

$$u = 0 \text{ on } \partial \Omega$$

on the domain $\Omega := \{(x,t) : t > 0, x > -t^{1/2}\}$. Our results imply that $u$ and $u_x$ are Hölder continuous up to the boundary if $f$ is and $f(0,0) = 0$. This example arises in the convexification of the nonlinear parabolic problem studied in MRC Technical Summary Report 02394.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.
Table of Contents

1. Statement of the result

2. The constant coefficient problem
   2.1 Auxiliary lemmas
   2.2 The Cauchy problem
   2.3 The boundary value problem

3. Extension to variable coefficients
   3.1 Reduction of the problem
   3.2 Auxiliary lemmas
   3.3 Proof of the main theorem
LINEAR PARABOLIC EQUATIONS WITH A SINGULAR LOWER ORDER COEFFICIENT

Klaus Höllig

1. Statement of the result

We consider the linear parabolic initial boundary value problem

\[
\begin{aligned}
    u_t(x,t) - a(x,t)u_{xx}(x,t) &= -t^{-1/2}b(x,t)u_x(x,t) + t^{-1/2}f(x,t), \\
    (x,t) &\in \Omega_T := [0,1] \times [0,T], \\
    u(x,0) &= \phi(x), x \in [0,1], \\
    u(v,t) &= \psi(t), t \in [0,T], v = 0,1.
\end{aligned}
\]

Problems of this type, with a singular coefficient of \( u_x \), may arise when transforming parabolic equations from a domain with curved boundaries to the standard domain \( \Omega_T \). Consider, e.g., the heat equation,

\[
\begin{aligned}
    v_t - \Delta v &= t^{-1/2}g \text{ on } \Omega \\
    v &= 0 \text{ on } \partial \Omega
\end{aligned}
\]

on the domain \( \Omega = \{(y,t) : 0 < y < 1 + t^{1/2}, t \in [0,T]\} \). By the change of variables \( x = y/(1 + t^{1/2}) \), (1.2) is equivalent to (1.1) with \( u(x,t) = v(y,t), f(x,t) = g(y,t), \phi = \psi = 0, a = (1 + t^{1/2})^{-2} \) and \( b(x,t) = \frac{1}{2}x/(1 + t^{1/2}) \). More generally, any parabolic equation on a domain with smooth vertical boundaries that touch the \( x \)-axis, but have nonzero curvature as \( t \to 0 \), leads, after a change of variables, to a problem of the form (1.1).

We shall show that, under the assumptions on the coefficients and the data specified in (1.3) - (1.5) below, the solution of problem (1.1) and its partial derivative with respect to \( x \) are Hölder continuous up to the
boundary. This is, in general, no longer true for a singularity of the form 
\( t^{-1/2-\varepsilon}, \varepsilon > 0 \), in the coefficient of \( u_x \).

We use the Hölder norms

\[
|\chi|_{a,I} := \sup_{z,z' \in I} |\chi(z) - \chi(z')|/|z - z'|^a
\]

\[
|\chi|_{a,I} := |\chi|_{a=1,I} + |\chi|_{a,I}
\]

\[
|w|_{a,x,\Omega} := \sup_{(x,t),(x',t) \in \Omega} |w(x,t) - w(x',t)|/|x-x'|^a
\]

\[
|w|_{\beta,t,\Omega} := \sup_{(x,t), (x',t') \in \Omega} |w(x,t) - w(x',t')|/|t-t'|^\beta
\]

\[
|w|_{a,\beta,\Omega} := |w|_{a,x,\Omega} + |w|_{\beta,t,\Omega}
\]

\[
|w|_{a,\beta,\Omega} := |w|_{a,\Omega} + |w|_{a,\Omega}
\]

where \( a, \beta \in (0,1) \) and denote by \( H^a(I) \) and \( H^a(\Omega) \) the Hölder classes

Corresponding to the norms \( ||_{a,I} \) and \( ||_{a,\beta,\Omega} \). The subscripts \( I \) and \( \Omega \) are omitted if the domain is clear from the context. We also need the subspaces \( H^a([0,T]) := \{ x \in H^a([0,T]) : x(0) = 0 \} \) and \( H^a,I_x([0,T]) := \{ x \in H^a([0,T]) : x(0,T) = 0, x \in I \} \). For simplicity of notation we write \( t^x \) and \( t^w \) for the functions \( t \mapsto t^x(t) \) and \( (x,t) \mapsto t^w(x,t) \).

We assume that the coefficients \( a, b \) and the data \( f, \phi, \psi \) for the problem (1.1) satisfy

\[
\begin{align*}
\left\{ \begin{array}{l}
\begin{align*}
& a, b \in H^\frac{a}{2}, a/2 \\
& a > a_0 > 0
\end{align*}
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
\begin{align*}
& f \in H^a, a/2 \\
& \phi, \psi \in H^a \\
& \psi, t^{1/2} \phi \in H^{a/2}
\end{align*}
\end{array} \right.
\end{align*}
\]

\[-2-\]
Theorem 1.1. Under the assumptions (1.3) - (1.5) the solution of the problem (1.1) satisfies

\[
lu_x, u_t + \frac{1}{2} u_{xx} \leq a, a/2, \Omega_T <
\]

\[
c \left[ \left\| a \right\|_{a/2, \Omega_T} + \left\| \phi \right\|_{a, [0,1]} \right] + \sum_{\nu=0}^{1} \left\| \psi_{\nu} t^{1/2} \psi_{\nu} \right\|_{a/2, [0, T]} \]
\]

where the constant \( c \) depends on \( a, a_0, \| a \|_{a/2, \Omega_T} \) and \( T \). Moreover we have

\[
\lim_{t \to 0} t^{1/2} u_{xx}(x, t) = 0, \ x \in [0,1].
\]

Theorem remains valid if we replace in problem (1.1) \( t^{-1/2} b(x, t) \) by \( t^{-\gamma} b(x, t) \) with \( \gamma < 1/2 \). In this case \( t^{-\gamma} b u_x \) can be regarded as a minor term and the result can be obtained from Theorem 1.1 by iteration. The Theorem also extends to quasilinear problems of the type (1.1) with \( a = a(x, t, u), b = b(x, t, u), f = f(x, t, u) \) if we assume, e.g., that \( a, b, f \) are \( C^1 \). We stated the result in the simplest setting to focus on its essential feature, the singularity in the coefficient of \( u_x \).

The difficult part of the proof of Theorem 1.1 is to show (1.6) for the constant coefficient problem on the domain \( \mathbb{R}^+ \times [0, T] \) (section 2.3). We then use a standard technique [L, pp. 295-341] to extend the result to the case of variable coefficients (section 3). To keep the report self contained, the proofs of estimates for the Cauchy problem (section 2.2) are included.
although they are similar to the corresponding estimates for the heat equation.

2. The constant coefficient problem

2.1. Auxiliary lemmas

We list in this section some elementary inequalities and properties of Hölder norms. In the sequel $c$ denotes generic constants which may depend on $n, \gamma, a_0, |a,b|, a, b/2$.

(2.1) For $z, j > 0$,
$$ z^j \exp(-z) < c \exp(-cz) . $$

(2.2) For $t > s > 0, z \in \mathbb{R}$,
$$ c + c|z|/(t-s)^{1/2} < |z + t^{1/2} - s^{1/2}|/(t-s)^{1/2} < c + c|z|/(t-s)^{1/2} . $$

(2.3) For $t, s > 0, j > 1$,
$$ \int_0^t s^{-j/2}(t-s)^{-j/2} \exp(-z/s) ds < cz^{(1-j)/2} . $$

Lemma 2.1. For $X \in H^2([0, T]), Y \in (0, 1)$,

$$ |X|_Y < c \sup_{j \in \mathbb{Z}} |X|_{[2^{-j}T, 2^{-j+1}T]} . $$

When estimating differences $X(t) - X(t')$ it will be sometimes convenient to assume that $|t-t'| < c \min(t, t')$. In view of the Lemma this is no loss of generality when estimating $|X|_Y$.

Lemma 2.2. Let $v(y, t) := u(x + ct^{1/2}, t)$, then

$$ |v|_{a/2, t} < c |u|_{a, a/2} . $$

Lemma 2.3. For $t^{1/2} X \in H^2([0, T]), Y \in (0, 1)$, the norms $|t^{1/2} X|_Y$ and $(\int_0^T (t^{1/2} |X|_Y)^2 dt)^{1/2}$ are equivalent.
An analogous version of this Lemma holds for $H^a, \alpha/2$.

**Proof.** Assume that $\|t^{1/2-\gamma} \chi\|_\infty + \sup_0^T \|t^{1/2} \chi\|_{Y,[t,T]} < 1$ and let $0 < s < t < T$. We have

$$|t^{1/2} \chi(t) - s^{1/2} \chi(s)| <$$

$$|(t^{1/2} - s^{1/2}) \chi(t)| + |s^{1/2} (\chi(t) - \chi(s))| <$$

$$(t-s)^{-1/2} t^{-1/2+\gamma} + (t-s)^{-\gamma} < 2(t-s)^{\gamma}$$

and therefore $|t^{1/2} \chi|_Y < 2$.

Now assume that $|\chi|_Y < 1$ with $\overline{\chi} := t^{1/2} \chi$ and let $0 < s/2 < t < s < T$. We have

$$|\chi(s) - \chi(t)| = |s^{-1/2} \chi(s) - t^{-1/2} \chi(t)| <$$

$$|s^{-1/2} (\chi(s) - \chi(t))| + |(s^{-1/2} - t^{-1/2}) \chi(t)| <$$

$$s^{-1/2} (s-t)^{\gamma} + t^{-3/2} (s-t)^{\gamma} <$$

$$2t^{-1/2} (s-t)^{\gamma}$$

and therefore $|\chi|_{Y,[t,T]} < ct^{-1/2}$.

**Lemma 2.4.** For $\gamma \in (0,1/2)$ define

$$\mathcal{Q}^\gamma := \{ \chi : \chi(0) = \lim_{t \to 0} t^{1/2} \chi'(t) = 0 \}.$$  

(2.6)

$$\|\chi\|_{\mathcal{Q}} := \|t^{1/2} \chi'\|_{Y,[0,T]} < \infty.$$

Then we have, for $\chi \in \mathcal{Q}^\gamma$,

(2.7)

$$|\chi|_{Y^{1/2}} \cdot |t^{-1/2} \chi|_Y < c \|\chi\|_{\mathcal{Q}}.$$
2.2. The Cauchy problem

In this section we consider the problem

\[
\begin{cases}
  u_t - au_{xx} - t^{-1/2} bu_x = t^{-1/2} f, \ (x,t) \in \mathbb{R} \times [0,T) \\
  u(x,0) = \phi(x), \ x \in \mathbb{R}
\end{cases}
\]

(2.8)

where \( a > 0 \) and \( b \) are constants.

Theorem 2.1. Let \( f \in H^{0,\alpha/2}(\mathbb{R} \times [0,T]), \ \phi, \phi' \in H^\alpha(\mathbb{R}). \) Then the solution of problem (2.8) satisfies

\[
\|u, u_x, t^{1/2} u_x, u_{xx}, \alpha/2 \| \leq c \{ \|f\|_{\alpha, \alpha/2} + \|\phi, \phi'\|_\alpha \}
\]

(2.9)

where \( c \) depends on \( \alpha \) and \( b. \) Moreover we have

\[
\lim_{t \to 0} t^{1/2} u_{xx}(x,t) = 0, \ x \in \mathbb{R}.
\]

(2.10)

The change of variables \( y := x + 2bt^{1/2}, \ v(y,t) := u(x,t), \ g(y,t) := f(x,t) \), transforms the problem (2.8) into the heat equation

\[
\begin{cases}
  v_t - a v_{yy} = t^{-1/2} g, \ (y,t) \in \mathbb{R} \times [0,T) \\
  v(y,0) = \phi(y), \ y \in \mathbb{R}
\end{cases}
\]

(2.11)

Therefore the analysis of the Cauchy problem is fairly simple. We merely have to take into account the singular behavior of \( t^{-1/2} g \) as \( t \to 0. \) However, for convenience of the reader, we give a complete proof of Theorem 2.1.

By Lemma 2.2 it suffices to prove the assertions (2.9) and (2.10) for the solution \( v \) of problem (2.11), i.e.

\[
\|v, v_y, t^{1/2} v_y, v_{yy}, \alpha/2 \| \leq c \{ \|g\|_{\alpha, \alpha/2} + \|\phi, \phi'\|_\alpha \}
\]

(2.12)

\[
\lim_{t \to 0} t^{1/2} v_{yy}(y,t) = 0, \ y \in \mathbb{R}.
\]

(2.13)
Denote by \( \Gamma(x,t) := (4\pi t)^{-1/2} \exp(-\frac{x^2}{4t}) \) the fundamental solution of the heat equation. We have the estimate [L, p. 274]

\[
|D^j_x D^k_t \Gamma(x,t)| < ct^{-j/2-k} \exp(-c \frac{x^2}{t}).
\]

Also note that

\[
\int_{ \mathbb{R}^n } D^j_x \Gamma(x,t) dx = \begin{cases} 1, & j=0 \\ 0, & j>0 \end{cases}.
\]

\[
\int_0^\infty \Gamma(x,t) dt = \frac{1}{2}.
\]

To prove (2.12) and (2.13) we assume that \( a = 1 \) in problem (2.11) and consider two cases

(i) For \( q \equiv 0 \) we have

\[
|v_y, v_{yy}, \phi, \phi'| \leq c |\phi'| \alpha
\]

\[
|v, v_{yy}, \phi, \phi'| \leq c |\phi'| \alpha
\]

By Lemma 2.3, this proves (2.12) once we show

\[
|v_{yy}(y,t)| = |\int_{ \mathbb{R}^n } \Gamma_y(y,z,t)(\phi'(z) - \phi'(y)) dz| \leq ct^{-1/2 + \alpha/2} |\phi'| \alpha.
\]

(ii) Now assume that \( \phi = 0 \).

Lemma 2.5.

\[
|v_y \phi, \phi_y| \leq c |\phi| \alpha_y.
\]

Proof. Assume that \( |g| \alpha_y < 1 \) and let \( y - y' = h > 0 \). Using (2.15) we ha
Estimating these terms, using (2.14), we get

\[
|I_1| < c \int \int (t-s)^{-1/2} \exp\left(-c \frac{(y-z)^2}{t-s}\right) \frac{1}{|y-s|^{\alpha}} \, ds \, dz
\]

\[
< c \int \int \frac{1}{|y-s|^{1+\alpha}} < c \alpha^a
\]

For the second inequality we have used (2.3) with \( j = 2 \). \( |I_2| \) is estimated similarly.

\[
|I_3| < h \int \Gamma_{yy}(\xi-z,t-s) |z-y|^{-1/2} |z-y|^{-\alpha} \, dz \, ds
\]

with \( \xi = \xi(z,s) \in (y',y) \). Since for \( |y-z| > 2h = 2(y-y') \), \( |y-z|/2 < |\xi-z| < 2|y-z| \), we obtain, using (2.3) with \( j = 3 \),

\[
|I_3| < ch \int \int (t-s)^{-3/2} \exp\left(-c \frac{(y-z)^2}{t-s}\right) \frac{1}{|y-z|^{\alpha}} \, ds \, dz
\]

\[
< ch \int \int \frac{1}{|y-z|^{-2+\alpha}} < ch^a
\]

\[
|I_4| < \int_0^t \int \Gamma(y-z,t-s) \frac{1}{|y-z|^{1/2}} \frac{1}{h} \, ds \, dz
\]

\[
< ch^a \int_0^t (t-s)^{-1/2} \frac{1}{s^{1/2}} \, ds < ch^a
\]

**Lemma 2.5.**

(2.18)

\[
|y|^{\alpha}/2 < c \alpha^{\alpha} y
\]
Proof. Assume that \( |g_{a,y} | < 1 \) and let \( t - t' = h \in (0, t/2) \). Using (2.15), we have

\[
\begin{align*}
\nu_y(y, t) - \nu_y(y, t') &= \\
&= \int_{t-2h}^{t} \int_0 \Gamma_y(y-z, t-s)s^{-1/2}(g(z, s) - g(y, s))dzds \\
&- \int_{t-2h}^{t'} \int_0 \Gamma_y(y-z, t'-s)s^{-1/2}(g(z, s) - g(y, s))dzds \\
&+ \int_{0}^{t-2h} \int_0 \Gamma_y(y-z, t-s) - \Gamma_y(y-z, t'-s)s^{-1/2}(g(z, s) - g(y, s))dzds \\
&= \sum_{\nu=1}^{3} \Gamma_{\nu}.
\end{align*}
\]

Estimating these terms, using (2.14), we get

\[
|I_1| < c \int_{t-2h}^{t} \exp(-c \frac{(y-z)^2}{t-s})s^{-1/2}|y-z| dzds
\]

\[< c \int_{t-2h}^{t} (t-s)^{-1/2+\alpha/2} s^{-1/2} ds < c \alpha/2.\]

\[|I_2| \text{ is estimated similarly.}\]

\[
|I_3| < c \int_{t-2h}^{t} \exp(-c \frac{(y-z)^2}{t-s})s^{-1/2}|y-z| dzds
\]

\[< c \int_{t-2h}^{t} (t-s)^{-3/2+\alpha/2} s^{-1/2} ds < c \alpha/2.\]

\[\text{Lemma 2.7.}\]

\[|t^{1/2-\alpha/2}v_{yy}| < c |g_{a,y}|.\]

Proof. Assuming \( |g_{a,y} | < 1 \) we have, using (2.14) and (2.15),

\[
|\nu_{yy}(y, t)| = |\int_{0}^{t} \int_0 \Gamma_{yy}(y-z, t-s)s^{-1/2}(g(z, s) - g(y, s))dzds|
\]

\[< c \int_{(t-s)^{-3/2}} \exp(-c \frac{(y-z)^2}{t-s})s^{-1/2}|y-z| dzds\]
\[ c \int_0^t (t-s)^{-1+\alpha/2} s^{-1/2} \, ds \leq ct^{-1/2+\alpha/2} \]

This Lemma proves (2.13). In combination with Lemma 2.6 and the fact that \( v(\cdot,0) = 0 \) it also shows that

\[ |v|_{1/2,t} \leq \|v\|_{1/2,\infty} \leq c \|g\|_{\alpha, \alpha/2} \]

and it follows from Lemma 2.5 that

\[ \|v, v\|_{\alpha, \alpha/2} \leq c \|g\|_{\alpha, \alpha/2} \]

To complete the proof of assertion (2.12), we note that

\[ |v_{yy}|_{\alpha, \alpha/2, \mathbb{R}\times[s,T]} \leq c(|v_{yy}(\cdot, s)|_{\alpha} + |t^{-1/2}g|_{\alpha, y, \mathbb{R}\times[s,T]}) \leq c(s^{-1/2}|v_{y}(\cdot, s/2)|_{\alpha} + s^{-1/2}|g|_{\alpha, y, \mathbb{R}\times[s/2,T]}) \leq c s^{-1/2}|g|_{\alpha, y} \]

In view of Lemmas 2.3 and 2.7 this yields

\[ \|v_{yy}\|_{\alpha, \alpha/2} \leq c \|g\|_{\alpha, y} \]

2.3. The boundary value problem

In this section we consider the problem

\[
\begin{cases}
  u_t - au_{xx} - t^{-1/2}bu_x = t^{-1/2}f, \ (x,t) \in \mathbb{R}_+ \times (0,T) \\
  u(x,0) = \phi(x), \ x \in \mathbb{R}_+ \\
  u(0,t) = \psi(t), \ t \in [0,T] 
\end{cases}
\]

(2.20)

where \( a > 0 \) and \( b \) are constants.
Theorem 2.2. Let \( f \in H^{a,a/2}(\mathbb{R}_+ \times [0,T]) \), \( \phi, \phi' \in H^{a}(\mathbb{R}_+) \), \( \psi, \psi' \in H^{a/2}([0,T]) \) and assume that the compatibility conditions

\[
\begin{align*}
\phi(0) &= \psi(0) \\
\lim_{t \to 0} t^{1/2} \psi(t) &= f(0,0) + b \phi'(0)
\end{align*}
\]

hold. Then the solution of problem (2.20) satisfies

\[
\|u, u_x, t^{1/2} u_{xx}\|_{a,a/2} < c(\|f\|_{a,a/2} + \|\phi, \phi'\|_{a} + \|\psi, t^{1/2} \psi'\|_{a/2})
\]

where \( c \) depends on \( a \) and \( b \). Moreover we have

\[
\lim_{t \to 0} t^{1/2} u_{xx}(x,t) = 0, \quad x \in \mathbb{R}_+.
\]

Denote by \( \tilde{f}, \tilde{\phi} \) smooth extensions of the function \( f, \phi \) to the domains \( \mathbb{R} \times [0,T] \) and \( \mathbb{R} \) respectively. Subtracting from \( u \) the solution of the Cauchy problem with right hand side \( t^{-1/2} \tilde{f} \) and initial values \( \tilde{\phi} \) and using Theorem 2.1, we see that we have to prove Theorem 2.2 only for \( f = \phi = 0 \), an assumption we make throughout this section. In this case, the compatibility conditions (2.21), together with the assumption \( \psi, t^{1/2} \psi' \in H^{a/2}([0,T]) \), can be stated in the form

\[
\psi \in Q^{a/2}
\]

where \( Q \) has been defined in (2.6). The change of variables \( y := x + 2bt^{1/2} \) is of no help for the proof of Theorem 2.2 since \( \mathbb{R}_+ \times [0,T] \) is transformed to the domain \( \{(y,t) : t \in [0,T], y > 2bt^{1/2}\} \). However, we may assume \( a = 1 \), by a linear change of the \( t \)-variable.

Let us first obtain a representation for the solution of problem (2.20) (with \( a = 1, f = \phi = 0 \)) in terms of the fundamental solution \( \Phi(x-y,t,s) \) for the equation

\[
u_t - u_{xx} - t^{-1/2} b u_x = 0.
\]
By taking Fourier-transforms we see that

\[ K(x, t, s) = \Gamma(x + 2b(t^{1/2} - s^{1/2}), t-s) \]

(2.26)

\[ = (4\pi)^{-1/2} (t-s)^{-1/2} \exp \left( -\frac{(x+2b(t^{1/2} - s^{1/2}))^2}{4(t-s)} \right). \]

Using (2.1), (2.2), (2.14) and the fact that \( K \) satisfies equation (2.25) for \( x, t-s \neq 0 \) one can easily check that \( K \) satisfies the same estimates as the fundamental solution of the heat equation

(2.27)

\[ |D_x^k K(x, t, s)| \leq c(t-s)^{-1/2-k} \exp \left( -c \frac{x^2}{t-s} \right). \]

Also note, that

(2.28) \[ \frac{\partial}{\partial s} K(x, t, s) + \Delta_x K(x, t, s) + \Delta_s K(x, t, s) = 0, \quad x, t-s \neq 0, \]

which follows from (2.26):

Proposition 2.1. Let \( \varphi \in \mathcal{O}_{\alpha/2} \). The solution of problem (2.20) with \( f = \psi = 0 \) can be represented in the form

(2.29)

\[ u(x, t) = -2 \int_0^t K_x(x, t, s) \varphi(s) ds \]

where \( \varphi \in \mathcal{O}_{\alpha/2} \) is the solution of

(2.30)

\[ \varphi(t) = \psi(t) + 2 \int_0^t K_x(0, t, s) \varphi(s) ds \]

and

(2.29a)

\[ 0 = u(0, t) + 2 \int_0^t K_x(0, t, s) \varphi(s) ds \]

(2.31)

\[ |t^{1/2} \varphi'|_{\alpha/2} \leq c |t^{1/2} \varphi|_{\alpha/2}. \]

\[ u(0, t) = \varphi(t) \]

Proof. We claim that the operator \( R \) defined by

(2.32)

\[ (Rx)(t) := -2 \int_0^t K_x(0, t, s) x(s) ds \]

is a strict contraction on the space \( \mathcal{O} \) with respect to the norm

\[ x_{\mathcal{O}} = |t^{1/2} \varphi'|_{\alpha/2}. \]

We have

\[ -2 K_x(0, t, s) = b(t^{1/2} - s^{1/2})^{1/2} \exp \left( -\frac{(b(t^{1/2} - s^{1/2}))^2}{t-s} \right) \]

and we rewrite \( Rx \) in the form

\[ K_x(x, t-s) = \frac{2 [(x+2b)(t-s) - b^2]}{b(t-s)} K(x, t-s). \]

-12-
(R_X)(t) = \int_0^1 b \frac{1}{\pi^{1/2}} \frac{1}{(1-s)^{1/2} (1+s)^{1/2}} \exp\left(-\frac{(b(1-s^{1/2})^2}{1-s}\right)x(ts)ds

=: \int_0^1 r(b,s)x(ts)ds.

Substituting \( z := (b(1-s^{1/2})^2)/(1-s) \) we see that

\[ l_1(t) = \int_0^1 \int_0^z \frac{2}{b^2} \frac{1}{b^2 + s} \exp(-z)dz < 1. \tag{2.33} \]

Since

\[ t^{1/2}(R_X)'(t) = \int_0^1 (r(b,s)s^{1/2})(ts)^{1/2}x'(ts)ds \]

this implies that \( R \) is a contraction on \( Q \).

It remains to show that \( u \), given by (2.29), satisfies the boundary condition \( u(0, \ast) = \psi \). We write (2.29) in the form

\[ u(x, t) = \int_0^t \frac{x}{2t^{1/2}(t-s)^{3/2}} \exp\left(-\frac{(x+2b(t^{1/2} - s^{1/2}))^2}{4(t-s)}\right)x(t)ds + \int_0^t \frac{x}{2t^{1/2}(t-s)^{3/2}} \exp\left(-\frac{(x+2b(t^{1/2} - s^{1/2}))^2}{4(t-s)}\right)(x(s) - x(t))ds + \int_0^t \frac{b(t^{1/2} - s^{1/2})}{2t^{1/2}(t-s)^{3/2}} \exp\left(-\frac{(x+2b(t^{1/2} - s^{1/2}))^2}{4(t-s)}\right)x(s)ds \]

\[ = \sum_{\nu=1}^3 I_\nu(x) \]

and obtain

\[ \lim_{x \to 0} I_1(x) = \]

\[ x(t) \lim_{x \to 0} \int_0^t \frac{x^2}{2s^{1/2}} s^{-3/2} \exp\left(-\frac{(1+2b((t/x)^{1/2} - (t/x-s)^{1/2}))/2}{4s}\right)ds = \]

\[ \left(\frac{x(2\sqrt{b}(t-s))}{x-x}\right)u(r, (s)) \]

\[ \left(\frac{t^{1/2} + b((t-x)/s)}{b-s}\right) \]
\[
x(t) \int_0^t \frac{1}{2t^{1/2}} s^{-3/2} \exp\left(-\frac{1}{4s}\right) ds = x(t).
\]

\[
|I_2(x)| < c \int_0^t x(t-s)^{-3/2} \exp\left(-c \frac{s^2}{t-s}\right) (t-s)^{q/2} |x|_{c/2} ds < cx^{q/2} |x|_{c/2}.
\]

\[
\lim_{t \to 0} I_3(x) = (Rx)(t) = \Phi(t) - x(t).
\]

We now analyse the smoothness of the solution of problem (2.20) via the representation (2.29).

**Proposition 2.2.** Let \( v \) be defined by (2.29) with \( x \in \mathcal{D}^\gamma \), \( \gamma < 1/2 \), then

\[
(2.34) \quad ||v||_{Y,t} < c ||x||_Y.
\]

**Proof.** Assume \( ||x||_Y < 1 \). In particular, since \( x(0) = 0 \), \( ||x(t)|| < t^\gamma \). We let \( t-t' = h \in (0,t/3) \) and write

\[
-\frac{1}{2} (u(x,t) - u(x,t')) =
\]

\[
\int_{t-2h}^t K_x(x,t,s)(x(s) - x(t)) ds
\]

\[
- \int_{t-2h}^{t'} K_x(x,t',s)(x(s) - x(t')) ds
\]

\[
+ \int_0^{t-2h} (K_x(x,t,s) - K_x(x,t',s))(x(s) - x(t')) ds
\]

\[
+ \int_0^{2h} K_x(x,t,t-s)(x(t) - x(t')) ds
\]

\[
+ \int_0^{2h} (K_x(x,t,t-s) - K_x(x,t',t'-s))(x(t')) ds
\]

\[
+ \int_0^{t-2h} (K_x(x,t,t-2h-s) - K_x(x,t',t'-2h-s))(x(t')) ds
\]

\[
+ \int_0^h K_x(x,t',h-s) x(t') ds
\]

\[
= \sum_{v=1}^7 I_v.
\]
We set
\[(2.35)\quad A(t,s) := \frac{x + 2b(t^{1/2} - s^{1/2})}{2(t-s)^{1/2}}\]

and with this abbreviation
\[K_x(x,t,s) = -\frac{1}{2^{1/2}} (t-s)^{-1} A(t,s) \exp(-A(t,s)^2)\]
\[= K_1(x,t,s) + K_2(x,t,s)\]

where
\[K_1(x,t,s) = -\frac{1}{4^{1/2}} \frac{x}{(t-s)^{3/2}} \exp(-A(t,s)^2)\]
\[K_2(x,t,s) = -\frac{1}{2^{1/2}} \frac{b}{(t-s)^{1/2} (t^{1/2} + s^{1/2})} \exp(-A(t,s)^2)\]

Note that, by (2.2),
\[(2.36)\quad \exp(-A(t,s)^2) < c \exp(-c \frac{x^2}{t-s})\]

Using this, \(t-t' = h < t/3\) and (2.27) we estimate the integrals \(I_y\) as follows.

\[|I_1| \leq c \int_{t-2h}^t (|x_1| + |x_2|)(t-s)^{1/2} ds\]
\[< c(h^2 + \int_{t-2h}^t (t-s)^{-1/2} (t-s)^{-1/2} ds) < ch^2.\]

\[|I_2| \text{ is estimated similarly.}\]
\[|I_3| \leq ch \int_0^{t-2h} |x_1(x,t,s)| (t-s)^{1/2} ds\]
\[< ch \int_0^{t-2h} (t-s)^{-1/2} ds < ch^2.\]
\[|I_4| \leq ch^2 \int_0^{t-2h} (|x_1| + |x_2|)\]
\[< ch^2 (1 + \int_0^{t-2h} t^{-1/2} ds) < ch^2.\]
Since \( \left| \frac{d}{dA} (A \exp(-A^2)) \right| \leq c \) and

\[
|A(t,t-s) - A(t',t'-s)| = \frac{s}{c s^{1/2} t^{1/2} + (t-s)^{1/2}} \frac{s}{c s^{1/2} t^{1/2} + (t'-s)^{1/2}} \leq c s^{1/2} t^{-1} h^{1/2}
\]

we obtain

\[
|I_5| \leq c(t')^{\gamma} \int_0^{2h} s^{-1} |A(t,t-s)\exp(-A(t,t-s)^2) - A(t',t'-s)\exp(-A(t',t'-s)^2)| ds
\]

\[
< ch^{1/2} t^{\gamma-1} \int_0^{2h} s^{-1/2} ds < ch^{\gamma}.
\]

Since

\[
|A(t,t-2h-s) - A(t',t'-2h-s)| =
\]

\[
c(s+2h)^{-1/2} |t^{1/2} - (t-2h-s)^{1/2} - (t'-2h-s)^{1/2}| < c(s+2h)^{-1/2} h(t'-2h-s)^{-1/2}
\]

we obtain, for \( \gamma < 1/2 \),

\[
|I_6| < c(t')^{\gamma} \int_0^{t'-2h} (s+2h)^{-1} |A(t,t-2h-s)\exp(-A(t,t-2h-s)^2) - A(t',t'-2h-s)\exp(-A(t',t'-2h-s)^2)| ds
\]

\[
< ct^{\gamma} \int_0^{t'-2h} (s+2h)^{-3/2} h(t'-2h-s)^{-1/2} ds
\]

\[
< ct^{\gamma} h^{1/2} t^{\gamma-1} < ch^{\gamma}.
\]

\[
|I_7| < c(t')^{\gamma} \int_0^{h} (t-h+s)^{-1} ds < ct^{\gamma} h t^{-1} < ch^{\gamma}.
\]

**Proposition 2.3.** Let \( u \) be defined by (2.29) with \( \chi \in Q^\gamma, \gamma < 1/2 \), then

\[
|u_x|_{\gamma,t} < c|t^{1/2} \chi'|_{\gamma}.
\]
Proof. Assume $|t^{1/2}x'|_\gamma < 1$. By (2.28) we have

$$u_x(x,t) = -2 \int_0^t K_{xx}(x,t,s)x(s)ds =$$

$$= -2 \int_0^t K(x,t,s)x'(s)ds + 2b \int_0^t K(x,t,s)s^{-1/2}x(s)ds$$

$$=: v(x,t) + w(x,t).$$

Since, by Lemma 2.4, $|t^{-1/2}x|_\gamma < c$, Proposition 2.2 implies $|w|_{\gamma,t} < c$. Let $t-t' =: h \in (0,t/3)$. As in the proof of Proposition 2.2 we write the difference $1/2(v(x,t) - v(x,t'))$ in the form $\sum_{v=1}^7 J_v$ where the integrals $J_v$ are defined as $I_v$ but with $K_x$ replaced by $K$ and $x$ by $x'$. Using $t-t' = h < t/3$, (2.27) and the inequalities (c.f. Lemma 2.3)

$$|x'(t)| < ct^{-1/2+\gamma}$$

$$|x'(t) - x'(s)| < c(t-s)^{\gamma-1/2}, s < t,$$ we estimate the integrals $J_v$ as follows.

$$|J_1| < c \int_{t-2h}^t (t-s)^{-1/2}(t-s)^{\gamma-1/2}ds < ch^\gamma.$$ $|J_2|$ is estimated similarly.

$$|J_3| < ch \int_0^{t-2h} |K_t(x,\xi,s)| (t-s)^{\gamma-1/2}ds$$

$$< ch \int_0^{t-2h} (t-s)^{-3/2+\gamma}s^{-1/2}ds < ch^\gamma.$$ $|J_4| < c(t')^{-1/2} \int_0^{2h} s^{-1/2}ds < ch^\gamma.$

With $A$ defined by (2.35) we have, using (2.37),

$$|J_5| < c(t')^{-1/2} \int_0^{2h} s^{-1/2}\exp(-A(t,t-s)^2) - \exp(-A(t',t'-s)^2)ds$$

$$< ct^{-1/2} \int_0^{2h} t^{-1}h^{1/2}ds < ch^\gamma.$$
By (2.38),

\[ |\exp(-\Lambda(t,t-2h-s)^2) - \exp(-\Lambda(t',t'-2h-s)^2)| \]

\[ < c(s+2h)^{-1/2}h(t'-2h-s)^{-1/2} |\Lambda(\xi,\xi-2h-s)\exp(-\Lambda(\xi,\xi-2h-s)^2)| \]

\[ < c(s+2h)^{-1/2}h(t'-2h-s)^{-1/2} \frac{\exp(-c \frac{x^2}{s+2h}) + (s+2h)^{1/2}(t')^{-1/2}}{(s+2h)^{1/2}} \]

It follows that

\[ |J_0| < c(t')^{-1/2+\gamma} \int_0^{t'-2h} (s+2h)^{-1/2} |\exp(-\Lambda(t,t-2h-s)^2)| ds \]

\[ < ct^{-1/2+\gamma} h \int_0^{t'-2h} \frac{x}{(s+2h)^{-3/2}} \exp(-c \frac{x^2}{s+2h})(t-2h-s)^{-1/2} ds \]

\[ + t^{-1/2} h (s+2h)^{-1/2} (t'-2h-s)^{-1/2} ds | \]

\[ < ct^{-1/2+\gamma} h t^{-1/2} < ch^\gamma . \]

\[ |J_0| < c(t')^{-1/2+\gamma} \int_0^{h} (t-h+s)^{-1/2} dx \]

\[ < ct^{-1/2+\gamma} h t^{-1/2} < ch^\gamma . \]

**Proposition 2.4.** For \( \psi \in \mathcal{Q}^{a/2} \), the solution of problem (2.20) (with \( f = \phi = 0 \)) satisfies

\[ |u_x| \leq c |t^{1/2} \psi|_{a/2} . \]

**Proof.** Assume \( |t^{1/2} \psi|_{a/2} < 1 \). Since \( v := u_x \) satisfies (2.25) and \( v(\cdot,0) = 0, v(0,\cdot) = u_x(0,\cdot) \) we have

\[ v(x,t) = -2 \int_0^t K_x(x,t,s) \bar{x}(s) ds \]

with \( \bar{x} \) the solution of

\[ \bar{x}(t) = u_x(0,t) + 2 \int_0^t K_x(0,t,s) \bar{x}(s) ds . \]

By Proposition 2.3, \( |u_x(0,\cdot)|_{a/2} < c \), and the proof of Proposition 2.1, in particular (2.33), shows that this implies \( |\bar{x}|_{a/2} < c \).
Let \( x - x' = h > 0 \) and assume that \( t > h^2 \). We write

\[
- \frac{1}{2} (v(x, t) - v(x', t)) =
\]

\[
\int_{t-h^2}^t K_x(x,t,s) \overline{\chi(s) - \chi(t)} ds
\]

\[
- \int_{t-h^2}^t K_x(x',t,s) \overline{\chi(s) - \chi(t)} ds
\]

\[
+ \int_0^{t-h^2} (K_x(x,t,s) - K_x(x',t,s)) \overline{\chi(s) - \chi(t)} ds
\]

\[
+ \int_0^t (K_x(x,t,s) - K_x(x',t,s)) \overline{\chi(t)} ds
\]

\[
= \frac{4}{\nu} I_v
\]

Using \( h^2 < t, (2.27) \) and \( |\overline{\chi(t)}| < ct^{\alpha/2} \) we estimate these integrals as follows.

\[
|I_1| < c \int_{t-h^2}^t (t-s)^{-\nu}(t-s)^{\alpha/2} ds < ch^\alpha.
\]

\[
|I_2| \text{ is estimated similarly.}
\]

\[
|I_3| < ch \int_0^{t-h^2} |K_{xx}(\xi,t,s)|(t-s)^{\alpha/2} ds
\]

\[
< ch \int_0^{t-h^2} (t-s)^{-3/2 + \alpha/2} ds < ch^\alpha.
\]

Set \( I(x) := \int_0^t K_x(x,t,s) ds \). By (2.29) we have

\[
|I'(x)| = |\int_0^t K_{xx}(x,t,s) ds |
\]

\[
< |K(x,t,*)|_0 t + c \int_0^t |K_x(x,t,s)| s^{-1/2} ds
\]

\[
< c(t^{-1/2} \int_0^t \frac{S}{(t-s)^{3/2}} \exp(-c \frac{t}{t-s}s^{-1/2} ds
\]

\[
+ \int_0^t \frac{1}{(t-s)^{1/2}(t^{1/2} + s^{1/2})} s^{-1/2} ds
\]

\[
< ct^{-1/2}.
\]