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Pseudodifferential operators and Hardy kernels on $L^p(\mathbb{R}^+)$


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Pseudodifferential Operators and Hardy Kernels on \( L^p(\mathbb{R}^+) \) (*).

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Introduction.

We develop an algebra of pseudodifferential operators on \( L^p(\mathbb{R}^+) \), \( 1 < p < \infty \), which includes \( H \), the Hilbert transform restricted to \( \mathbb{R}^+ \), and some classical Hardy kernels. Such operators arise in the study of singular integral equations in \( L^q(\mathbb{R}^+) \) since \( H^2 = -I + K \), where \( K \) is a Hardy kernel operator. The algebra of operators described here, called Mellin operators in \( OP\Sigma_{1/p} \), are defined via the Mellin transform. As in the case of the Hilbert transform [Sh 1] or a Hardy kernel [FJL 1], the spectrum of the operator depends upon the \( L^p \) space on which it acts. As described in the remarks of Section 5, there are operators in \( OP\Sigma_{1/\nu} \) for all \( p \), \( 1 < p < \infty \), which admit a parametrix in \( OP\Sigma_{1/\nu} \) for some values of \( p \), but do not have a parametrix in \( OP\Sigma_{1/\nu} \) for other values of \( p \); the parametrices in \( OP\Sigma_{1/\nu} \) for different values of \( p \), do not necessarily agree on \( C_0^\infty(\mathbb{R}^+) \).

E. Shamir [Sh 1, Sh 2] studied the spectrum of the Hilbert transform on \( L^p(\mathbb{R}^+) \). G. I. Šekin [E 1, E 2] has made an extensive study of operators defined via the Mellin transform and given applications to weighted \( L^2 \) spaces and boundary value problems. In [N] J. Nourrigat has defined a class of pseudodifferential operators on \( \mathbb{R}^+ \) defined by the Mellin transform and studied their properties on weighted \( L^2 \) spaces. B. A. Plamenevskii in [P] has studied an algebra of pseudodifferential operators in \( \mathbb{R}^+ \times S^{n-1} \) defined using the Mellin transform. H. O. Cordes and E. A. Herman [CH] studied singular integrals on \( L^p(\mathbb{R}^+) \).

In Section 1 we state the properties of the Mellin transform and Mellin multipliers to be used in the sequel. A representation for variable symbol Mellin operators is studied in Section 2. The space of symbols, \( \Sigma_{1/p} \), and

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the space of Mellin operators, $OP\Sigma_{1/\rho}$, are defined in Section 3; the principal symbol $\sigma_p$ is defined. The symbolic calculus is developed in Section 4. The operators in $OP\Sigma_{1/\rho}$ which admit a parametrix in $OP\Sigma_{1/\rho}$ are characterized by their symbols in Theorem 5; the remarks following Theorem 5 describe a typical situation. The index of an elliptic operator in $OP\Sigma_{1/\rho}$ is studied in Section 6. An application to an oblique derivative problem for Laplace’s equation in a plane sector is given in Section 7.

1. – Preliminaries on the Mellin transform.

We shall deal with functions in $L^p = L^p(\mathbb{R}^+)$, $1 < p < \infty$, with the norm

$$\|f\|_p = \left(\int_0^\infty |f(x)|^p dx\right)^{1/p}.$$ 

It will be convenient to consider functions $g(x) \in L^p_\nu(\mathbb{R}^+)$ where

$$\|g\|_{\nu,\nu} = \left(\int_0^\infty |g(x)|^\nu dx\right)^{1/\nu}.$$ 

If $f(x) \in L^p$, we define $f_p(x) = x^{1/p}f(x) \in L^p_\nu(\mathbb{R}^+)$ and the functions $F(u) = f(\exp[-u])$, and $F_x(u) = f_x(\exp[-u])$, $u \in \mathbb{R}$. Note that

$$\|f\|_{\nu,\nu} = \|f_p\|_{L^\nu(\mathbb{R})} = \left(\int_{-\infty}^{+\infty} \exp[-u]\|F(u)\|^\nu du\right)^{1/\nu}.$$ 

If $f(x) \in C^\infty_0(\mathbb{R}^+)$ we define the Fourier transform of $F_p$ as the function

$$\hat{F}_p(\xi) = \int_{-\infty}^{+\infty} \exp[-iu\xi]F_p(u) du = \int_0^{+\infty} x^{1/p+it}f(x) dx, \quad \xi \in \mathbb{R}.$$ 

The Mellin transform of a function $f \in C^\infty_0(\mathbb{R}^+)$ is defined as

$$\tilde{f}(z) = \int_0^\infty x^{z-1}f(x) dx, \quad z \in \mathbb{C}.$$
It follows that for \( f \in C_0^\infty(\mathbb{R}^+) \), \( \tilde{f}(z) \) is an entire function and we have the inversion formula

\[
\tilde{f}(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^{-z} \tilde{f}(z) \, dz ,
\]

where the notation \( \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \ldots dz \) denotes contour integration along the path \( z = a + i\xi, \ -\infty < \xi < \infty \). Integration by parts shows that

\[
\left( -x \frac{d}{dx} \right)^\sim (z) = z\tilde{f}(z) \quad \text{and} \quad \left( (\log x)\tilde{f} \right)^\sim (z) = \frac{d}{dz} \tilde{f}(z) .
\]

For \( \tau \) real, \( \delta > 0 \), we use the notation

\[
\mathcal{S}_{\tau,\delta} = \{ z \in \mathbb{C} : \tau - \delta < \Re z < \tau + \delta \} .
\]

If \( f \) is measurable on \( \mathbb{R}^+ \) and the integral (1.2) is absolutely convergent for all \( z \) in some strip \( \mathcal{S}_{\tau,\delta} \) we shall call the integral \( \tilde{f}(z) \) the Mellin transform of \( f \); under these conditions \( \tilde{f}(z) \) is a holomorphic function in \( \mathcal{S}_{\tau,\delta} \). We make the following definition.

**Definition 1.** Let \( b(z) \) be a bounded measurable function on the line \( \Re z = 1/p \). Then we say \( b \) is a Mellin multiplier on \( L^p \) iff the map

\[
Bf(x) = \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} x^{-z} b(z) \tilde{f}(z) \, dz , \quad x > 0 , \quad f \in C_0^\infty(\mathbb{R}^+) ,
\]

is extendable as a bounded linear operator on \( L^p \).

By (1.1), \( \tilde{f}_x(\xi) = \tilde{f}(1/p + i\xi) \), so that \( b \) is a Mellin multiplier on \( L^p \) iff the function \( \xi \to b(1/p + i\xi) \) is a Fourier multiplier on \( L^p(\mathbb{R}) [H] \).

We give the following examples which will be essential ingredients in the algebra of operators to be constructed in Section 3.

1) The Hilbert transform on \( L^p(\mathbb{R}^+) \). The Hilbert transform of a function \( f \in L^p \) is defined as

\[
Hf(x) = \text{p.v.} \frac{1}{\pi} \int_0^\infty \frac{f(y)}{x-y} \, dy .
\]
Following Shamir [Sh 1] and Eskin [E 1] we can represent $H$ as

$$Hf(x) = \frac{1}{2\pi i} \int \frac{1}{1 - \exp[2\pi iz]} f(z) dz, \quad f \in C^\infty_0(\mathbb{R}^+) .$$

It is well known that $H$ is a bounded operator on $L^p$. Define the function

$$\theta(z) = \frac{1}{1 - \exp[2\pi iz]} = \frac{1}{2} \left( 1 + \frac{1 + \exp[2\pi iz]}{1 - \exp[2\pi iz]} \right).$$

Then $\theta(z)$ is a Mellin multiplier on $L^p$, $1 < p < \infty$.

2) Hardy kernels on $L^p$. Let $k(x)$ be a measurable function on $\mathbb{R}^+$ such that for some $a$, $b$ with $0 < a < b < 1$,

$$\int_0^\infty x^{a-1} |k(x)| dx + \int_0^\infty x^{b-1} |k(x)| dx < \infty .$$

Then for all $p$, $a < 1/p < b$, the Hardy operator with kernel $k$ is defined on $L^p$ by

$$Kf(x) = \int_0^\infty k\left(\frac{x}{y}\right) f(y) \frac{dy}{y} .$$

Following [FJL 1], for $f \in C^\infty_0(\mathbb{R}^+)$,

$$Kf(x) = \frac{1}{2\pi i} \int \frac{1}{1 - \exp[2\pi iz]} \tilde{k}(z) \tilde{f}(z) dz ,$$

where $\tilde{k}(z)$ is the Mellin transform of the kernel $k$ which is defined and holomorphic for $a < \text{Re } z < b$.

3) The operator $T_{\zeta,p}$ (a particular Hardy kernel). Let $1 < p < \infty$ and $\zeta \in \mathbb{C}$, $\text{Re } \zeta \neq 1/p$. For $f \in C^\infty_0(\mathbb{R}^+)$, define

$$T_{\zeta,p}f(x) = \frac{1}{2\pi i} \int \frac{1}{1 - \exp[2\pi iz]} \frac{1}{\zeta - z} \tilde{f}(z) dz .$$

The function $b(z) = 1/(\zeta - z)$ is a Mellin multiplier on $L^p$, $1/p \neq \text{Re } \zeta$, \ldots
and the $L^p$ norm of the operator $T_{ζ,p}$ is bounded by $C |\text{Re } ζ - 1/p|^{-1}$ ($C$ is independent of $p$). If $1/p < \text{Re } ζ$, the kernel for $T_{ζ,p}$ on $L^p$ is given by

$$k_{ζ,p}(x) = \begin{cases} 0, & 0 < x < 1, \\ x^{-ζ}, & x > 1. \end{cases}$$

If $\text{Re } ζ < 1/p$, the kernel for $T_{ζ,p}$ on $L^p$ is given by

$$k_{ζ,p}(x) = \begin{cases} -x^{-ζ}, & 0 < x < 1, \\ 0, & x > 1. \end{cases}$$

The bound for the $L^p$ norm of $T_{ζ,p}$ is a consequence of Young’s inequality.

2. – A class of bounded operators on $L^p(\mathbb{R}^+)$.

We now introduce a class of Mellin integral operators on $L^p(\mathbb{R}^+)$ with variable kernels.

**THEOREM 1.** Let $a(\delta, z)$ be a function defined for $\delta > 0$ and $z$ in some strip $S_{1/p, p}, 1 < p < \infty$. Suppose that for all $\delta$, $a(\delta, z)$ is holomorphic in $S_{1/p, p}$ and that there is an $C > 0$ and a constant $C$ such that

$$\sup_{z > 0} |a(\delta, z)| < C(1 + |z|)^{-1-\varepsilon}, \quad z \in S_{1/p, p}.$$

Then the operator defined by

$$(2.6) \quad Af(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-z}a(\delta, z)f(z)dz, \quad f \in C_0^\infty(\mathbb{R}^+),$$

is extendable as a bounded operator on $L^p$.

**PROOF.** Let $0 < \delta < \delta$ and let $\Gamma_{1/p, \delta}$ denote the contour

$$\left\{ \frac{1}{p} + \delta_1 + i\xi, -\infty < \xi < \infty \right\} \cup \left\{ \frac{1}{p} - \delta_1 - i\xi, -\infty < \xi < \infty \right\}.$$
Using this representation for \( a(x, z) \) in (2.6) and applying Fubini’s Theorem, we obtain

\[
Af(x) = \frac{1}{2\pi i} \int_{(1)} a(x, \zeta) \frac{1}{\zeta - z} f(z) dz = \frac{1}{2\pi i} \int_{(1)} a(x, \zeta) T_{\zeta, x} f(x) d\zeta.
\]

An application of Minkowski’s Integral Inequality gives

\[
\|Af\|_p \leq \frac{1}{2\pi} \int_{(1)} \sup_{z > 0} |a(x, z)| \|T_{\zeta, x} f\|_p |d\zeta| < C \|f\|_p. \tag{2.7}
\]

**THEOREM 2.** Let \( a(x, z) \) be a function defined for \( x > 0 \) and for \( z \) in some strip \( S_{1/p, \delta} \). Suppose that

1) \( a(x, z) \) is continuously differentiable in \( R^+ \times S_{1/p, \delta} \) and holomorphic in \( z \),

2) For some \( \epsilon > 0 \) there is a constant \( C \) such that for all \( x \) and \( z \)

\[
|a(x, z)| \leq C \left( \frac{x}{1 + x^2} \right)^\epsilon \left( \frac{1}{1 + |z|^{1+\epsilon}} \right), \tag{2.8}
\]

Then the operator \( A \) defined by (2.6) is compact on \( L^p \).

**PROOF.** By the proof of Theorem 1 and (2.7) it follows that the operators \( f \to \chi_{(0, \beta)}(x) A f(x) \) and \( f \to \chi_{(1, \infty)}(x) A f(x) \) have small \( L^p \) norm if \( \beta \) is small. The map \( f \to -x(d/dx) A f(x) = Tf(x) \) is represented by

\[
Tf(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-z} \left\{ a(x, z) z - x \frac{d}{dx} a(x, z) \right\} f(z) dz.
\]

By (2.8) and Theorem 1, \( T \) is bounded on \( L^p \). From these observations it follows that the family \( \{Af: \|f\|_p \leq 1\} \) is equicontinuous in \( L^p(0, N) \) for every \( N \) and that

\[
\lim_{N \to \infty} \sup_{\|f\|_p \leq 1} \int_{|x| > N} |Af(x)|^p dx = 0.
\]

This establishes the compactness of \( A \) on \( L^p \). \( \text{ q.e.d.} \)
3. Spaces of symbols and Mellin operators.

As a preliminary step we introduce some spaces of functions and their Mellin transforms.

**Definition 2.** Let $\tau$ be real. If $\delta > 0$, by $\mathcal{F}_{\tau,\delta}$ we denote the class of functions $a(x) \in C^\infty(R^+)$ such that the following property holds: for every $\delta_1, 0 < \delta_1 < \delta$, and every $j$ there is a constant $C = C(\delta_1, j, a)$ such that

$$\left| \frac{d^j}{dx^j} a(x) \right| \leq C x^{-\tau} \left( \frac{x}{1 + x^2} \right)^{\delta_1}.$$ 

By $\mathcal{F}_{\tau}$ we denote the space of functions $a(x)$ such that $a \in \mathcal{F}_{\tau,\delta}$ for some $\delta$.

**Definition 3.** Let $\tau$ be real. If $\delta > 0$, by $\mathcal{H}_{\tau,\delta}$ we denote the class of functions $b(z)$ which are defined and holomorphic in the strip $S_{\tau,\delta}$ and such that the following property holds: for every $\delta_1, 0 < \delta_1 < \delta$, and every $j$ and $k$ there is a constant $C = C(\delta_1, j, k, b)$ such that

$$\left| \frac{d^j}{dz^j} b(z) \right| \leq C$$

for all $z \in S_{\tau,\delta}$. By $\mathcal{H}_{\tau}$ we denote the space of functions $b(z)$ such that $b \in \mathcal{H}_{\tau,\delta}$ for some $\delta$.

The fact that the functions in $\mathcal{H}_{\tau}$ are precisely the Mellin transforms of the functions in $\mathcal{F}_{\tau}$ is consequence of the following result whose proof is contained in the article of A. Avantaggiati [A, Sec. 2].

**Lemma 1.** If $a \in \mathcal{F}_{\tau,\delta}$, then its Mellin transform $\hat{a} \in \mathcal{H}_{\tau,\delta}$. Conversely, given $b \in \mathcal{H}_{\tau,\delta}$, define the function

$$a(x) = \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} x^{-s} b(z) dz, \quad x > 0.$$ 

Then $a \in \mathcal{F}_{\tau,\delta}$ and $\hat{a} = b$.

We define the symbols of the class of smoothing operators to be constructed.

**Definition 4.** Let $\tau$ be real. If $\varepsilon, \delta > 0$, by $\Phi_{\tau,\delta,\varepsilon}$ we denote the class of functions such that:

1) $a(x, z) \in C^\infty(R^+ \times S_{\tau,\delta})$, 

2) $\hat{a}(x, z) \in C^\infty(R^+ \times S_{\tau,\delta})$, 

3) $a(x, z) = a(z, x)$, 

4) $a(x, z) = a(z, x)$, 

5) $a(x, z) = a(z, x)$, 

6) $a(x, z) = a(z, x)$, 

7) $a(x, z) = a(z, x)$, 

8) $a(x, z) = a(z, x)$, 

9) $a(x, z) = a(z, x)$, 

10) $a(x, z) = a(z, x)$.
2) for all \( x \), \( a(x, z) \) defines a holomorphic function on \( S_{\varepsilon, \delta} \),

3) for each \( \varepsilon_1 \), \( 0 < \varepsilon_1 < \varepsilon \), and each \( \delta_1 \), \( 0 < \delta_1 < \delta \), and each \( M \), \( j \), \( k \) there is a constant \( C = C(\varepsilon_1, \delta_1, M, j, k, a) \) such that

\[
|x^M \left(x \frac{\partial}{\partial x}\right)^j \left(z \frac{\partial}{\partial z}\right)^k a(x, z)| < C \left(\frac{x}{1 + x^2}\right)^{\varepsilon_1},
\]

for \( z \in S_{\varepsilon, \delta} \).

By \( \Phi \) we denote the class of functions \( a(x, z) \) which belong to \( \Phi_{\varepsilon, \delta} \) for some \( \varepsilon, \delta \).

We recall that the function \( \theta(z) = 1/(1 - \exp[2\pi i z]) \) is holomorphic in the strip \( 0 < \text{Re}\, z < 1 \) and that for all \( M \) and \( j \), uniformly in the strip \( 0 < \delta < \text{Re}\, z < 1 - \delta < 1 \),

\[
\lim_{\text{Im}\, z \to \infty} \left| z^M \frac{d^j}{dz^j} (\theta(z) - 1) \right| = 0
\]

and

\[
\lim_{\text{Im}\, z \to -\infty} \left| z^M \frac{d^j}{dz^j} \theta(z) \right| = 0.
\]

It follows that \( \theta(z)(1 - \theta(z)) \in \mathcal{F}_{1/p} \) for \( 1 < p < \infty \).

Finally we are ready for the definition of the space of symbols of an algebra of Mellin operators on \( L^p(\mathbb{R}^+) \).

**Definition 5.** Let \( 1 < p < \infty \). Denote by \( \Sigma_{1/p} \) the space of functions \( a(x, z) \in C^\infty(\mathbb{R}^+ \times S_{1/p, \delta}) \) for some \( \delta = \delta(a) > 0 \) and for which there is a representation of the following form in \( \mathbb{R}^+ \times S_{1/p, \delta} \):

\[
a(x, z) = a_+(x)\theta(z) + a(x)(1 - \theta(z)) + a(z) + a(x, z)
\]

where

1) \( a_+(x) \) and \( a_-(x) \) are extendable as continuous functions on \( \mathbb{R}^+ \) in such a way that \( a_+(x) - a_-(0) \in \mathcal{F}_a \),
2) \( a(z) \in \mathcal{F}_{1/p} \),
3) \( a(x, z) \in \Phi_{1/p} \).

**Definition 6.** For each symbol \( a \in \Sigma_{1/p} \) we define the Mellin operator

\[
Af(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-z} a(x, z) f(z) dz, \quad f \in C^\infty_0(\mathbb{R}^+).
\]
The space of all such operators will be denoted by $OP\Sigma_{1/p}$. The function $a(x, z) \in \Sigma_{1/p}$ will be called the symbol of the Mellin operator $A$. If the symbol of the operator $A$ is also in the class $\Phi_{1/p}$, we shall write $A \in OP\Phi_{1/p}$ and shall call $A$ a smoothing operator.

**Definition 7.** If $A$ is a Mellin operator with symbol

$$a(x, z) = a_+(x)\theta(z) + a_-(x)(1 - \theta(z)) + a(x) + \alpha(x, z),$$

the function $a(x, z) - \alpha(x, z)$ will be called the principal symbol of $A$ and be denoted by $\sigma_\phi(A)(x, z)$.

From Theorem 1 it follows that if $A \in OP\Sigma_{1/p}$, then $A$ can be extended as a bounded operator on $L^p$; moreover, if $\sigma_\phi(A)(x, z) \equiv 0$, $A$ is a compact operator on $L^p$.

4. - The symbolic calculus for $OP\Sigma_{1/p}$.

We study the compositions and adjoints of operators in $OP\Sigma_{1/p}$.

**Theorem 3.** Let $A, B \in OP\Sigma_{1/p}$. Then $AB \in OP\Sigma_{1/p}$. Moreover, if

$$\sigma_\phi(A)(x, z) = a_+(x)\theta(z) + a_-(x)(1 - \theta(z)) + a(x)$$

and

$$\sigma_\phi(B)(x, z) = b_+(x)\theta(z) + b_-(x)(1 - \theta(z)) + b(x),$$

then

$$\sigma_\phi(AB)(x, z) = c_+(x)\theta(z) + c_-(x)(1 - \theta(z)) + c(x),$$

where

$$\begin{align*}
(4.1) & \quad c_+(x) = a_+(x)b_+(x) \\
(4.2) & \quad c_-(x) = a_-(x)b_-(x) \\
(4.3) & \quad c(x) = \sigma_\phi(A)(0, z) \cdot \sigma_\phi(B)(0, z) - a_+(0)b_+(0)\theta(z) - a_-(0)b_-(0)(1 - \theta(z)).
\end{align*}$$

**Proof.** We shall first show that the composition of two smoothing operators is a smoothing operator. Let $a(x, z), b(x, z) \in \Phi_{1/p}$ and let $A$ and $B$
be the corresponding Mellin operators. For \( f \in C^\infty_0(\mathbb{R}^+) \),

\[
Bf(z) = \lim_{\alpha \to \infty} \frac{1}{2\pi i} \int_{1-\alpha}^{1+\alpha} x^{-w} b(x, w) \tilde{f}(w) \, dw \, dx
\]

where \( \tilde{b}(z, w) \) denotes the Mellin transform of \( b(x, w) \) in the \( x \)-variable. Thus

\[
\int_{1-\alpha}^{1+\alpha} x^{-w} c(x, w) \tilde{f}(w) \, dw \, dx
\]

In the above calculations, the absolute convergence of the integrals justifies the use of Fubini’s Theorem.

To verify that \( c(x, w) \in \Phi_{1/p} \) we apply repeatedly the observation that if \( a(x, z) \in \Phi_{1/p} \), then\( b(x, z) \in \Phi_{1/p} \).

Let the symbol of \( A \) be

\[
a(x, z) = a_+ (x) \theta (z) + a_- (x) (1 - \theta (z)) + a(z)
\]

and let \( b(x, z) \in \Phi_{1/p} \).

In this case we argue as above and again use Fubini’s Theorem to obtain that

\[
ABf(x) = \frac{1}{2\pi i} \int_{1-\alpha}^{1+\alpha} x^{-w} c(x, w) \tilde{f}(w) \, dw \, dx
\]
where
\[ c(x, w) = \frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-\sigma} a(x, v + w) \tilde{b}(v, w) dv. \]

Consider, e.g., the contribution of
\[ \frac{a_+(0)}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-\sigma} \theta(v + w) \tilde{b}(v, w) dv. \]

Modulo \( \Phi_{1/p} \), this integral is \( c_+(x, w) \) where

\[ c_+(x, w) = \frac{a_+(0)}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-\sigma} \theta(v + w) \tilde{b}(v, w) dv, \]

Now
\[ c_+(x, w) = \frac{a_+(0)}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-\sigma} \int_0^1 \frac{\partial}{\partial w} (w + tv) \tilde{b}(v, w) dt dv. \]

To show that \( c_+(x, w) \in \Phi_{1/p} \), we observe that for some \( \delta > 0 \), it is possible to shift the integral \( \int_{0-i\infty}^{0+i\infty} \ldots dv \) to either of the integrals

\[ \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \ldots dv \quad \text{or} \quad \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \ldots dv. \]

Hence \( x^{-\sigma} w^N (\partial/\partial x)^l (\partial/\partial \omega)^k c_+(x, w) \) is a linear combination of integrals of the form

\[ I(x, w) = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} x^{-(\sigma+\delta)\log x} \int_0^1 t^n (w + tv)^M (w + tv) \tilde{b}(v, w) dt dv, \]

where \( b_+(x, w) \in \Phi_{1/p} \). The integrals \( I(x, w) \) are absolutely convergent and for \( w \) in some strip around \( \text{Re } w = 1/p \) are bounded by \( C |\log x|^s \). Repeating the argument with \( \delta \) replaced by \(-\delta\) shows that \( c_-(x, w) \in \Phi_{1/p} \). In the same manner one shows that

\[ \frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-\sigma} \{ a_-(x) (1 - \theta(v + w)) + a(v + w) \} \tilde{b}(v, w) dv \]

gives a function in \( \Phi_{1/p} \).
The next step is to show that if \( A \in \text{OP}_{\Phi_1/2} \) and \( B \in \text{OP}_{\Sigma_{1/2}} \), then \( AB \in \text{OP}_{\Phi_{1/2}} \). Let \( a(x, z) \) be the symbol of \( A \) and suppose that \( b(x, z) = b_+(x)\theta(z) + b_-(x)(1 - \theta(z)) + b(z) \) is the symbol of \( B \). Denote by \( B_+ \) the operator

\[
B_+ f(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-v} b_+(x)\theta(z)f(z)\,dz.
\]

We have that

\[
AB_+ f(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-v} a(x, w)\tilde{f}(w)\,dw + C_1 f(x)
\]

where

\[
c(x, w) = \frac{\theta(w)}{2\pi i} \int_{0 - i\infty}^{0 + i\infty} x^{-v} a(x, v + w)(b_+(x) - b_+(0))^{-}(v)\,dv
\]

and \( C_1 \) is a smoothing operator. Since \( (b_-(x) - b_+(0))^{-}(v) \in \mathcal{F}_\theta \), it may be shown that \( c(x, w) \in \Phi_{1/2} \). The other terms in the composition can be handled similarly.

As the next step we consider two operators \( A_+, B_+ \) with symbols \( a_+(x)\theta(z) \) and \( b_+(x)\theta(z) \). Let \( \Theta \) be the operator with symbol \( \theta(z) \). Then

\[
A_+B_+ = a_+(x)b_+(0)\Theta^2 + a_+(x)(\Theta b_+(x)\Theta)
\]

where

\[
b_+(x) = b_+(x) - b_+(0) \in \mathcal{F}_\theta.
\]

The second part can be written as

\[
Cf(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-v} a(x, w)\tilde{f}(w)\,dw,
\]

where

\[
c(x, w) = a_+(x)\theta(w) \frac{1}{2\pi i} \int_{0 - i\infty}^{0 + i\infty} x^{-v}\theta(v + w)\tilde{b}_+(v)\,dv.
\]

Using the argument for the integral (4.4), we have that modulo a symbol in \( \Phi_{1/2} \), \( c(x, w) = a_+(x)b_+(x)\theta^2(w) \).

At this point we observe that \( \theta^2 = \theta(\theta - 1) + \theta \) so that the operator \( \theta^2 \) contains the Hardy kernel operator with symbol \( \theta(z)(\theta(z) - 1) \in \mathcal{F}_{1/2} \). Thus
the principal symbol of $A_+B_+$ is given by
\[\sigma_\delta(A_+B_+)(x, z) = a_+(x)b_+(x)\theta(z) + a_+(0)b_+(0)\theta(z)(\theta(z) - 1).\]

Since the composition of two Hardy kernel operators with symbols in $\mathcal{F}_{1/p}$ is a Hardy kernel operator with symbol in $\mathcal{F}_{1/p}$, we leave to the reader the proof of the cases not considered explicitly and the calculation of the principal symbol. \(\Box\)

**Corollary.** If $A, B \in OP\Sigma_{1/p}$ the commutator
\[[A, B] = AB - BA \in OP\Phi_{1/p}.\]

We consider the adjoint of an operator $A \in OP\Sigma_{1/p}$. If $1/p + 1/q = 1$ we define $A^*: L^q \to L^q$ to be the operator such that
\[\int_0^\infty Af(x)\overline{g(x)}\,dx = \int_0^\infty f(x)A^*\overline{g(x)}\,dx, \quad f, g \in C_0^\infty(\mathbb{R}^+).\]

**Theorem 4.** Let $A \in OP\Sigma_{1/p}$ and $1/p + 1/q = 1$. Then $A^* \in OP\Sigma_{1/q}$; moreover, the principal symbol of $A^*$ is
\[
\sigma_\delta(A^*)(x, z) = \overline{\sigma_\delta(A)(x, 1-\overline{z})}.
\]
In particular if
\[
\sigma_\delta(A)(x, z) = a_+(x)\theta(z) + a_-(x)(1 - \theta(z)) + a(x),
\]
then
\[
\sigma_\delta(A^*)(x, z) = \overline{a_+(x)\theta(z)} + \overline{a_-(x)(1 - \theta(z))} + \overline{a(1-\overline{z})},
\]
Re $z$ near $1/q$.

**Proof.** We recall that the Hilbert transform $H$ is representable as $H = i(2\Theta - 1) \in OP\Sigma_{1/p}$. Using the kernel representation (1.3) of $H$, we have that $H^* = -H \in OP\Sigma_{1/q}$. A calculation shows that
\[
\sigma_\delta(H^*)(x) = \overline{\sigma_\delta(H)(1-\overline{z})}.
\]
Next consider the operator $Af(x) = a(x)f(x)$ where $a(x) - a(0) \in \mathcal{F}_\delta$. Then $A^*g(x) = \overline{a}(x)g(x)$. Representing $\Theta$ in terms of $H$ and applying Theorem 3
proves the theorem for operators of the form
\[ A = a_\times(x)\Theta + a_\times(x)(1 - \Theta). \]

We now consider an operator \( A \) which is a Hardy operator with symbol \( a(z) \in \mathcal{F}_{1/p} \). There is a kernel \( k(x) \in \mathcal{F}_{1/p} \) with \( \tilde{k} = a \) such that \( A f(x) = \int_0^\infty k(x,y) f(y) (dy/y) \). The adjoint \( A^* \) is representable with a kernel \( k^*(x) = (1/x)\tilde{k}(1/x) \). For \( \Re w \) near \( 1/q \) we have that \( \tilde{k}^*(w) = a(1 - \overline{w}) \in \mathcal{F}_{1/q} \).

Finally we show that if \( A \in OP\mathcal{F}_{1/p} \), then \( A^* \in OP\mathcal{F}_{1/q} \). If the symbol of \( A \) is \( a(x,z) \in \mathcal{F}_{1/q} \) and \( f, g \in C_0^\infty(\mathbb{R}^+) \), represent

\[
f(y) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} y^{-z} \tilde{f}(z) \, dz \quad \text{and} \quad g(y) = \frac{1}{2\pi i} \int_{1/q - i\infty}^{1/q + i\infty} y^{-z} \tilde{g}(z) \, dw.
\]

Using Fubini’s Theorem, we obtain that

\[
\int_0^\infty A f(y) g(y) \, dy = \int_0^\infty f(x) \frac{1}{2\pi i} \int_{1/q - i\infty}^{1/q + i\infty} x^{-z} c(x,w) \tilde{g}(w) \, dw \, dx,
\]

where

\[
c(x,w) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} \int_0^\infty x^{-z+1} a(y,z) \, dy \, dz.
\]

Performing the change of variables \( z \to 1 - \overline{w} + v, \Re v = 0 \), we have that

\[
c(x,w) = \frac{1}{2\pi i} \int_{0 - i\infty}^{0 + i\infty} x^{-v} \tilde{a}(v, (1 - \overline{w}) - v) \, dv.
\]

Using arguments similar to those for (4.4) and (4.5), we can show that \( c(x,w) \in \mathcal{F}_{1/q} \).

**Remark.** If \( A \in OP\mathcal{F}_{1/p} \), the transposed operator \( A^* \) is defined so that

\[
\int_0^\infty A f(x) g(x) \, dx = \int_0^\infty f(x) A^* g(x) \, dx, \quad f, g \in C_0^\infty(\mathbb{R}^+) .
\]
Then if $1/p + 1/q = 1$, $A \in OP_{1/q}$. If $\sigma_x(A)$ is given by (4.6) then

$$\sigma_x(tA)(x, z) = a_+(x)(1 - \theta(x)) + a_-(x)\theta(z) + a(1 - z),$$

Re $z$ near $1/q$.

REMARK. We observe that smoothing operators map $L^p$ into $\mathcal{F}_{1/p}$.

**Lemma 2.** Let $A \in OP_{1/p}$. Then if $f \in L^p$, $Af \in \mathcal{F}_{1/p}$.

**Proof.** Let the symbol of $A$ be $a(x, z) \in \mathcal{F}_{1/p}$. Define the function $k(x, t)$ by

$$k(x, t) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} t^{-z}a(x, z)dz.$$  

Then for some $\delta > 0$ and each $i, j$ there is a $C = C(\delta, i, j, k)$ such that

$$(4.9) \quad \left| \left( x \frac{\partial}{\partial x} \right)^i \left( t \frac{\partial}{\partial t} \right)^j k(x, t) \right| \leq Ct^{-1/p} \left( \frac{x}{1 + x^2} \right)^\delta \left( \frac{t}{1 + t^2} \right)^\delta.$$

Fix $\xi > 0$ and for $f \in C_0^\infty(\mathbb{R}^+)$ let

$$A_{\xi}f(x) = \int_0^\infty k \left( \xi, \frac{x}{y} \right) f(y) \frac{dy}{y}.$$

The Mellin transform of $A_{\xi}f$ is $a(\xi, z)f(z) \in \mathcal{F}_{1/p}$. Hence

$$A_{\xi}f(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-z}a(\xi, z)f(z)dz.$$

Putting $\xi = x$ we then have the representation

$$Af(x) = \int_0^\infty k \left( x, \frac{x}{y} \right) f(y) \frac{dy}{y} = \int y^{-1/p} k \left( x, \frac{x}{y} \right) \left[ y^{1/p}f(y) \right] \frac{dy}{y}.$$
It follows that \( x^{1/p}(x(d/dx))^j Af(x) \) is a linear combination of integrals of the form

\[
I(x) = \int_0^\infty \left( \frac{x}{y} \right)^{1/p} k_r \left( \frac{x}{y} \right) \left( \frac{y^{1/q}f(y)}{y} \right) \frac{dy}{y},
\]

where \( k_r(x, t) \) satisfies estimates of the form (4.9). By Hölder’s inequality,

\[
|I(x)| \leq C \left( \frac{x}{1 + x^2} \right)^\theta \left( \int_0^\infty \left( \frac{t}{1 + t^2} \right)^{4/5} dt \right)^{1/5} \|f\|_p,
\]

where \( 1/p + 1/q = 1 \). q.e.d.

5. Elliptic operators in \( OPΣ_{1/p} \).

We characterize the operators \( A \in OPΣ_{1/p} \) which are elliptic, i.e., for which there exists a parametrix \( B \in OPΣ_{1/p} \) such that \( AB - I \) and \( BA - I \) are smoothing operators.

**Theorem 5.** Let \( A \in OPΣ_{1/p} \) with principal symbol

\[
σ_p(A)(x, z) = a_+(x)θ(z) + a_-(x)(1 - θ(z)) + a(z).
\]

The following two conditions are equivalent:

1) There is an operator \( B \in OPΣ_{1/p} \) such that \( AB - I \in OPΦ_{1/p} \).

2) The following three conditions are satisfied by \( σ_p(A)(x, z) \):

\[
\begin{align*}
&\inf_{ξ ∈ R} \left| σ_p(A) \left( 0, \frac{1}{p} + iξ \right) \right| > 0, \\
&\inf_{z > 0} |a_+(x)| > 0, \\
&\inf_{z > 0} |a_-(x)| > 0.
\end{align*}
\]

**Proof.** Suppose that 1. is satisfied and let

\[
σ_p(B)(x, z) = b_+(x)θ(z) + b_-(x)(1 - θ(z)) + b(z).
\]

By Theorem 3

\[
σ_p(AB)(x, z) = 1 = 1 \cdot θ(z) + 1(1 - θ(z)).
\]
By (4.1) and (4.2) and the observation that
\[
\sigma_p(AB)(0, z) = \sigma_p(A)(0, z) \cdot \sigma_p(B)(0, z),
\]
we have the identities
\[
1 = a_+(x)b_-(x),
1 = a_-(x)b_+(x),
1 = \sigma_p(A)(0, z)\sigma_p(B)(0, z).
\]
Condition 2 follows.

Conversely, suppose that 2. is satisfied. Note that for some $\delta > 0$,
\[
\inf_{x \in \mathcal{S}_{1/p, \delta}} |\sigma_p(A)(0, z)| > 0.
\]

We define an operator $B$ with symbol
\[
b(x, z) = \frac{1}{a_+(x)}\theta(z) + \frac{1}{a_-(x)}(1 - \theta(z)) + b(z),
\]
where
\[
b(z) = \frac{1}{\sigma_p(A)(0, z)} - \frac{1}{a_+(0)}\theta(z) - \frac{1}{a_-(0)}(1 - \theta(z)),
\]
z $\in \mathcal{S}_{1/p, \delta}$. It may be shown, using the properties of $a_+(x)$, $a_-(x)$ and (3.1), (3.2), that $b(x, z) \in \Sigma_{1/p}$. A direct calculation using (4.1), (4.2), and (4.3) shows that $B$ is a parametrix for $A$. q.e.d.

**Definition 8.** If $A \in \text{OP}\Sigma_{1/p}$ and $A$ satisfies condition 1. or 2. of Theorem 5, we shall say that $A$ is an elliptic operator in $\text{OP}\Sigma_{1/p}$.

**Remark.** We emphasize that the definition of ellipticity in $\text{OP}\Sigma_{1/p}$ depends on $p$. The following situation is typical.

Consider an operator $A$ with principal symbol
\[
\sigma_p(A)(x, z) = a(x, z) = a_+(x)\theta(z) + a_-(x)(1 - \theta(z)) + a(z).
\]
Suppose that $\inf |a_+(x)| > 0$, $\inf |a_-(x)| > 0$, and that $a(z) \in \mathcal{F}_{1/p}$ for all $p$, $1 < p < \infty$. Then the function $\psi(z) = (a(0, z))^{-1}$ is meromorphic in the strip $0 < \Re z < 1$; moreover, in any strip $0 < \delta < \Re z < 1 - \delta < 1$, $\psi(z)$ has only a finite number of poles. Hence for all $p$ outside a discrete set, $\mathcal{N}(A)$, $A$ is elliptic in $\text{OP}\Sigma_{1/p}$. Define $b(x, z)$ by (5.2) where $b(z)$ is defined
by (5.3). Then $b(z)$ has the same poles and residues as $\varphi(z)$. Then if $p \notin \mathcal{N}(A)$, $b(x, z)$ is the principal symbol of a parametrix for $A$ in $OP\Sigma_{1/p}$. If $p_1$ and $p_2 \notin \mathcal{N}(A)$, let $B_{p_i} \in OP\Sigma_{1/p_i}$ be parametrices for $A$ in $OP\Sigma_{1/p_i}$, $i = 1, 2$. The calculation in [FJL 1] shows that for $f \in C^\infty_0(\mathbb{R}^+)$,

$$
B_{p_i} f(x) - B_{p_i} f(x) = \int_{0}^{\infty} L\left(\frac{x}{y}\right) f(y) \frac{dy}{y},
$$

where the kernel $k$ is given by

$$
k(x) = \frac{1}{2\pi i} \int_{1/p_1 - i\infty}^{1/p_1 + i\infty} x^{-s} b(z) dz - \frac{1}{2\pi i} \int_{1/p_2 - i\infty}^{1/p_2 + i\infty} x^{-s} b(z) dz.
$$

If $a(0, \zeta) = 0$ for some $\zeta$, $\operatorname{Re} \zeta$ between $1/p_1$ and $1/p_2$, then $k(x) \neq 0$ (see [FJL 1]).

**Remark.** If $A$ is elliptic in $OP\Sigma_{1/p}$ and $Af = 0$, $f \in L^p$, then $f \in \mathcal{F}_{1/p}$. This follows from Lemma 2.

### 6. The index of an elliptic operator in $OP\Sigma_{1/p}$

We will relate the index of an elliptic operator in $OP\Sigma_{1/p}$ to the winding numbers of the coefficients $a_\pm(x)$.

**Lemma 3.** For $v$ an integer define

$$
\varphi_v(x) = \exp\left(\frac{x}{x + 1}\right).
$$

Then

1) $\varphi_v : \mathbb{R}^+ \to S^1 = \{ |z| = 1 \}$ and the winding number of $\varphi_v$ is $v$.

2) $\varphi_v(x) - 1 \in \mathcal{F}_v$.

3) If $a(x)$ is a function mapping $\mathbb{R}^+ \to \mathbb{C} \setminus \{0\}$ such that $a(x) - a(0) \in \mathcal{F}_v$, and the winding number of $a$ is $v$, then there is a continuous homotopy $F : [0, 1] \times \mathbb{R}^+ \to \mathbb{C} \setminus \{0\}$ such that

(i) $F(0, x) = a(0) \varphi_v(x),$

(ii) $F(1, x) = a(x),$

(iii) $F(t, 0) = a(0)$, $0 < t < 1,$

(iv) $F(t, x) - a(0) \in \mathcal{F}_v$, $0 < t < 1.$
PROOF. Parts 1 and 2 follow by a calculation. To prove 3, we remark that if we replace \( a(x) \) by \( a(x)q_{-}(x) \) it is sufficient to construct the homotopy when \( v = 0 \). In this case it is well known that there is a homotopy \( G(t, x) \) which satisfies (i)-(iii) and such that \( G(t, \cdot) \in C^{\infty}(\mathbb{R}^+) \) for every \( t \). Let \( \varepsilon = \frac{1}{2} |a(0)| > 0 \) and choose \( \delta > 0 \) such that

\[
|G(t, x) - a(0)| < \varepsilon \quad \text{for} \quad 0 < t < 1, \quad 0 < x < 4\delta \quad \text{or} \quad (4\delta)^{-1} < x < \infty .
\]

Construct a nonnegative partition of unity on \( \mathbb{R}^+ \) as

\[
1 = \alpha_1(x) + \alpha_2(x) + \alpha_3(x)
\]

where \( \alpha_1(x) = 1 \) if \( 0 < x < \delta \), \( \alpha_2(x) = 0 \) if \( x > 2\delta \), \( \alpha_3(x) = 1 \) for \( x > \delta \), \( \alpha_3(x) = 0 \) for \( x < \frac{1}{2}\delta \) and \( \alpha_2(x) = 1 - \alpha_1(x) - \alpha_3(x) \). Define the homotopy \( F \) as

\[
F(t, x) = (\alpha_1(x) + \alpha_3(x))\left( a(0) + t(a(x) - a(0)) \right) + \alpha_2(x)G(t, x).
\]

The verification of (i)-(iv) is left to the reader. q.e.d.

As an application of the previous lemma we have the following theorem.

THEOREM 6. Let \( A \) be elliptic in \( OP\Sigma_{1/\nu} \) and let

\[
\sigma_\nu(A)(x, z) = a_+(x)\theta(z) + a_-(x)(1 - \theta(z)) + a(z).
\]

Suppose that \( v_+ \) and \( v_- \) are the winding numbers of \( a_+ \) and \( a_- \). Then \( A \) as an operator on \( L^\mu \) has index \( v = v_+ - v_- \).

The proof of Theorem 6 is accomplished by a sequence of lemmas.

LEMMA 4. With the notation of Theorem 6, \( A \) has the same index as the operator \( A^\# \) with symbol

\[
a^\#(x, z) = a_+(0)q_{+}(x)\theta(z) + a_-(0)q_{-}(x)(1 - \theta(z)) + a(z).
\]

PROOF. By Lemma 3 there are homotopies \( F_{\pm}(t, x) \) which connect \( a_{\pm}(x) \) to the functions \( a_{\pm}(0)q_{\pm}(x) \) in such a way that the operators \( A_t \) with symbols

\[
a_t(x, z) = F_{+}(t, x)\theta(z) + F_-(t, x)(1 - \theta(z)) + a(z)
\]

are elliptic in \( OP\Sigma_{1/\nu} \). Then \( A_0 = A \) and \( A_1 = A \mod OP\Phi_{1/\nu} \). Hence index \( A^\# = \text{index } A \) in \( L^\mu \). q.e.d.

LEMMA 5. With the notation of Theorem 6 and Lemma 4, the operator \( A \) has the same index as the operator \( A_\nu \) with symbol

\[
a_\nu(x, z) = q_{\nu}(x)\theta(z) + (1 - \theta(z)).
\]
PROOF. Let $A_0$ be the operator with symbol $a_0^s(z) = a^s(0, z)$ and $B_0$ be the operator with symbol $b_0(z) = (a_0^s(z))^{-1}$. Then $B_0A_0^s = I = A_0B_0$ so that $A_0$ has the same index on $L^p$ as the operator $B_0A_0^s = I + B_0(A_0^s - A_0^s_0)$. Since $\sigma_p(A_0^s - A_0^s_0)(0, z) = 0$, Theorem 3 yields that

$$\sigma_p(B_0A_0^s) = \varphi_{\gamma_+}(x)(\theta(z)) + \varphi_{\gamma_-}(x)(1 - \theta(z)).$$

Then index $(A_0) = \text{index } \varphi_{\gamma_-}, B_0A = \text{index } (A_0^s)$. q.e.d.

**Proof of Theorem 6.** It remains to calculate the index of $A_0$ on $L^p$. Since $\sigma_p(A_0)(0, z) = 1$, $A_0$ is elliptic in $OP\Sigma_{1/r}$, for all $r$, $1 < r < \infty$. If $f \in L^r$, $A_0f = 0$, then $f \in \mathcal{H}_{1/r}$, $1 < r < \infty$. If $g \in L^q, 1/p + 1/q = 1$, $A_0^s g = 0$, then $g \in \mathcal{H}_{1/r}$, $1 < r < \infty$. Thus the index of $A_0$ on $L^p$ is the index of $A_0$ on $L^1$. As an operator on $L^1$, $A_0$ is in the algebra considered by Cordes and Herman [CH], and its symbol $\sigma_{A_0}$, as defined in [CH], has winding number $v$ and hence index $v$. (The particular operator considered in [CH] was $K_0 = \Theta + [(\log x - 2i)/(\log x + 2i)](I - \Theta)$). q.e.d.

7. - Application to an oblique derivative problem in a plane sector.

Operators in $OP\Sigma_{1/r}$ arise naturally in the oblique derivative problem in a plane sector.

Let $\Omega = \{(x, y) \in \mathbb{R}^2; x > 0, y > 0\}$. We seek a solution of the following problem:

$$\begin{cases}
A u = 0 & \text{in } \Omega \\
\lim_{x \to 0^+} \left( \alpha_1(x) \frac{\partial u}{\partial y}(x, y) + \beta_1(x) \frac{\partial u}{\partial x}(x, y) \right) = \gamma_1(x) \in L^p(\mathbb{R}^1), \\
\lim_{x \to 0^+} \left( \alpha_2(y) \frac{\partial u}{\partial x}(x, y) + \beta_2(y) \frac{\partial u}{\partial y}(x, y) \right) = \gamma_2(y) \in L^p(\mathbb{R}^1),
\end{cases}
$$

where $\alpha_i$ and $\beta_i$ are real functions such that $\alpha_1^2(t) + \beta_1^2(t) = 1$.

If $\Phi_1(t), \Phi_2(t) \in C_0^\infty(\mathbb{R}^1)$ we study the single layer potential with density $\Phi_1$ along the $x$-axis and density $\Phi_2$ along the positive $y$-axis, namely,

$$u(x, y) = \frac{1}{2\pi} \int_0^\infty \log((x - t)^2 + y^2) \Phi_1(t) dt + \frac{1}{2\pi} \int_0^\infty \log(x^2 + (y - t)^2) \Phi_2(t) dt$$

$$= u_1(x, y) + u_2(x, y).$$
Then it is known [St, FJL2] that in $L^p(\mathbb{R}^+)$,

$$
\lim_{y \to 0^+} \frac{\partial u_1}{\partial y}(x, y) = \phi_1(x)
$$

$$
\lim_{y \to 0^+} \frac{\partial u_1}{\partial x}(x, y) = p. x \cdot \frac{1}{\pi} \int_0^\infty \frac{1}{x - t} \phi_1(t) \, dt = H \phi_1(x),
$$

$$
\lim_{y \to 0^-} \frac{\partial u_1}{\partial x}(x, y) = -\frac{1}{\pi} \int_0^\infty \frac{t}{y^2 + t^2} \phi_1(t) \, dt = -K_n \phi_1(y),
$$

$$
\lim_{y \to 0^-} \frac{\partial u_1}{\partial y}(x, y) = \frac{1}{\pi} \int_0^\infty \frac{y}{y^2 + t^2} \phi_1(t) \, dt = K \phi_1(y).
$$

Note that $K_n$ and $K$ are Hardy kernel operators with kernels

$$
k_n(t) = \frac{1}{\pi} \frac{1}{1 + t^2} \quad \text{and} \quad k(t) = \frac{1}{\pi} \frac{t}{1 + t^2}.
$$

Since $k_n(t)$ and $k(t) \in \mathcal{F}_1$, $1 < p < \infty$, we have that $K_n$ and $K$ are operators in $OPC^1_{\mathcal{V}}$ and that

$$
\sigma_n(K_n)(z) = \frac{-\cos((\pi/2)z)}{\sin(\pi z)} \quad \text{and} \quad \sigma_n(K_n)(z) = \frac{\sin((\pi/2)z)}{\sin(\pi z)}.
$$

Recall that the symbol of the operator $H$ may be written as

$$
\sigma(H)(z) = \frac{-\cos(\pi z)}{\sin(\pi z)}.
$$

Similar formulas hold for the boundary values of the gradient of $u_2$.

We make the following assumptions on the coefficients of the boundary operators in (7.1):

$$
\alpha_j(t) - \alpha_j(0), \quad \beta_j(t) - \beta_j(0) \in \mathcal{F}, \quad j = 1, 2.
$$

Then the boundary operators applied to $u$ give functions $\psi_1(t), \psi_2(t) \in L^p(\mathbb{R}^+)$ where

$$
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix} =
\begin{pmatrix}
\alpha_1(t) I + \beta_1(t) H & -\alpha_1(t) K_n + \beta_1(t) K
\\
-\alpha_2(t) K_n + \beta_2(t) K & \alpha_2(t) I + \beta_2(t) H
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}.
$$
We write the system (7.2) as \( \hat{\psi} = A\hat{\eta} \) where \( A \) is a matrix of operators in \( OP_{1/p} \). The matrix of principal symbols is given by

\[
\sigma_p(A)(t, z) = \begin{pmatrix}
\alpha_1(0) \cos \left( \frac{\pi}{2} z \right) + \beta_1(0) \sin \left( \frac{\pi}{2} z \right) \\
\sin(\pi z) \\

- \alpha_2(0) \cos \left( \frac{\pi}{2} z \right) + \beta_2(0) \sin \left( \frac{\pi}{2} z \right) \\
\sin(\pi z)
\end{pmatrix}
\]

where \( \alpha_j(t) = \alpha_j(t) + i\beta_j(t) \).

**Theorem 7.** For \( j = 1, 2 \), let \( \nu_j \) be the winding numbers of \( v_j(t) \). Suppose that

\[
\inf_{Re z = \frac{1}{p}} |\det \sigma_p(A)(0, z)| > 0.
\]

Then

1) There is a matrix \( B \) of operators in \( OP_{1/p} \) such that \( AB = I \) and \( BA = I \) are matrices of smoothing operators.

2) As an operator on \( L^p \times L^p \), the index of \( A \) is \( 2\nu_1 + 2\nu_2 \).

**Proof.** Since \( \alpha_1^2 + \beta_1^2 = 1 \), the function \( \det \sigma_p(A)(t, z) \) is the symbol of an elliptic operator \( D \) in \( OP_{1/p} \). Denote by \( E \) a parametrix for \( D \) and let \( B \) be the matrix of operators

\[
B = \begin{pmatrix}
E & 0 \\
0 & E
\end{pmatrix}
\begin{pmatrix}
\alpha_1(t) I + \beta_1(t) H & \alpha_1(0) K_0 - \beta_1(0) K_1 \\
\alpha_2(t) K_0 - \beta_2(0) K_1 & \alpha_1(t) I + \beta_1(t) H
\end{pmatrix}.
\]

The symbolic calculus establishes that, modulo a matrix of functions in \( \Phi_{1/p} \), \( \sigma_p(A)(t, z) \cdot \sigma_p(B)(t, z) = I \) and the first conclusion is established.

Let \( A_0 \) be the matrix of operators whose symbol is \( \sigma_p(A)(0, z) \) and let \( B_0 \) be the matrix of operators whose symbol is \( \sigma_p(B)(0, z) = [\sigma_p(A_0)(z)]^{-1}. \) Then \( B_0 A_0 = A_0 B_0 = I \) on \( L^p \times L^p \) so that the index of \( A \) is the index of \( B_0 A = I - B_0 (A - A_0) \). The matrix of principal symbols of \( B_0 A \) is

\[
\sigma_p(B_0 A)(t, z) = \begin{pmatrix}
\nu_1(t) \\
\nu_1(0)
\end{pmatrix}
\begin{pmatrix}
\alpha_1(t) \cos \left( \frac{\pi}{2} z \right) + \beta_1(0) \sin \left( \frac{\pi}{2} z \right) \\
\sin(\pi z) \\
- \alpha_2(t) \cos \left( \frac{\pi}{2} z \right) + \beta_2(0) \sin \left( \frac{\pi}{2} z \right) \\
\sin(\pi z)
\end{pmatrix}
\]

By Theorem 6, the index of \( B_0 A \) is \( 2\nu_1 + 2\nu_2 \).
Remark. For the operator $A_0$ there are always values of $p$ for which $\det \sigma_p(A_0)(z) = 0$ for some $z$, $\Re z = 1/p$.

Suppose that $\alpha_j(0) + i \beta_j(0) = \cos \gamma_j + i \sin \gamma_j$, $j = 1, 2$. Another representation of $\sigma_p(A_0)(z)$ is

$$\sigma_p(A_0)(z) = \frac{1}{\sin(\pi z)} \begin{pmatrix} \sin(\pi z - \gamma_1) & \sin \left( \frac{\pi}{2} z - \left( \frac{\pi}{2} - \gamma_1 \right) \right) \\ \sin \left( \frac{\pi}{2} z - \left( \frac{\pi}{2} - \gamma_2 \right) \right) & \sin(\pi z - \gamma_2) \end{pmatrix}.$$

Then $\det \sigma_p(A_0)(z) = 0$ when $z = (2k + 1)/3$ or $z = (2/\pi)(\gamma_1 + \gamma_2) + (2k + 1)$. In particular $\det \sigma_p(A_0)(1/3) = 0$, and $A_0$ is not a Fredholm operator on $L^3 \times L^3$.

This is in accordance with the results of [FJL 2] for double layer potentials for the Dirichlet problem for which the operators were not Fredholm for $p = \frac{3}{2}$.

REFERENCES


