Harmonic music “noise” signals between red (Brownian) and white noise

Brownian noise or red noise is the kind of signal noise produced by Brownian motion. A Brownian motion (i.e. a Wiener process) is a continuous stationary stochastic process having independent increments, i.e. \( B(t) - B(0) \) is a normal random variable with mean \( \mu \) and variance \( \sigma^2 t \), \( \mu, \sigma^2 \) constant real numbers. The density function of a Brownian motion is given by

\[
f_{B(t)}(x) = \frac{1}{\sqrt{2\pi \sigma^2 t}} e^{\frac{-(x-\mu)^2}{2\sigma^2 t}}.
\]

The sample paths of Brownian motion are not differentiable, a mathematical fact explaining the highly irregular motions of small particles. The total variation of Brownian motion over a finite interval \([0, T]\) is infinite. It holds

\[
\text{Var} \left[ \frac{B(t)}{t} \right] = \frac{\sigma^2}{t}.
\]

If \( W(t) \) is a Wiener process on the interval \([0, \infty)\), then, as well the process

\[
W^* (t) := \begin{cases} 
  dW(1/t), & t > 0 \\
  0 & , t = 0
\end{cases},
\]

White noise can be defined as the derivative of a Brownian motion (i.e. a Wiener process) in the framework of infinite dimensional distribution theory, as the derivative \( B'(t) \) of \( B(t) \), does not exist in the ordinary sense. Not only \( B'(t) \), but also all derivatives of Brownian motion are generalized functions on the same space. For each \( t \), the white noise \( B'(t) \) is defined as a generalized function (distribution) on an infinite dimensional space. A Brownian motion is obtained as the integral of a white noise signal \( dB(t) \), i.e.

\[
B(t) = \int_0^t dB(t)
\]

meaning that Brownian motion is the integral of the white noise \( dB(t) \) whose power spectral density is flat

\[
S_0 = |\text{Fourier}[B']\omega|^2 = \text{const}.
\]

This means, that the spectral density \( S_0 \) for white noise is flat, i.e. \( S_0 / \omega^5 = c \) i.e. it is inversely proportional to \( \omega^5 \). It holds \( \text{Fourier}[B']\omega = i\omega \text{Fourier}[B]\omega \). Therefore the power spectrum of Brownian noise is given by

\[
S(\omega) = |\text{Fourier}[B]\omega|^2 = \frac{S_0}{\omega^5}.
\]

This means, that the spectral density of Brownian (red) noise is \( S_0 / \omega^5 \), i.e. it is inversely proportional to \( \omega^5 \), meaning it has more energy at lower frequencies, even more so than pink noise.

The above indicates a spectral density for harmonic music “noise” signals given by

\[
S'(\omega) = |\text{Fourier}[B^*]\omega|^2 = \frac{S_0}{\omega}.
\]