On the lowest eigenvalue of a pseudo-differential operator

( sharp Gårding inequalities/uncertainty principle/subelliptic estimates/commutators of vector fields)

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ABSTRACT Positive lower bounds for pseudo-differential operators with nonnegative symbols are derived; the bounds in particular yield subelliptic estimates for operators arising as sums of squares of vector fields.

Let \( p(x,\xi) \) be a nonnegative symbol satisfying the estimates

\[ |\partial_x^\alpha \partial_{\xi}^\beta p(x,\xi)| \leq C_{\alpha\beta} M^{-|\alpha|} \]

We shall outline an algorithm to determine the order of magnitude of the lowest eigenvalue of the corresponding pseudo-differential operator \( p(x,D) \). This is closely related to earlier work on conditions ensuring the estimate

\[ \text{Re}(p(x,D)u,u) + C|u|^2 \geq 0 \quad u \in L^2(\mathbb{R}^n). \]

The sharpest known sufficient conditions for inequality 2 are the following:

(i) \( p \in S^2(\mathbb{R}^n \times \mathbb{R}^n), \quad p \geq 0 \) (see ref. 1)

(ii) \( p \in S^{6/5}(\mathbb{R}^n \times \mathbb{R}^n), \quad p + Tr^+p \geq 0 \), in which

\( Tr^+p \) is a nonnegative quantity defined in terms of the Hessian of \( p \) [see Hörmander (2) and also Melin (3)].

Our first main result on the eigenvalue problem, motivated by the uncertainty principle of quantum mechanics, is the following:

- Let \( Q_0 = \{ (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n ; |x|, |\xi| \leq 1 \}; \) say that a canonical transformation \( \Phi : (x,\xi) \rightarrow (y,\eta) \) mapping \( Q_0 \) into \( \mathbb{R}^n \times \mathbb{R}^n \) is a testing map if \( y - y_0 \) and \( (\eta - \eta_0)/M \) are \( C^\alpha \) functions of \( (x,\xi) \) with norms bounded by a fixed constant. Here \( (y_0,\eta_0) \) denotes \( \Phi(0,0) \), and \( \alpha \) is a constant that depends on \( \epsilon \) below.

**THEOREM 1.** If \( p(x,\xi) \geq 0 \) satisfies inequality 1, and \( K \geq C p^2 \) is a constant such that

\[ \|p^\alpha \Phi \|_{C(Q_0)} \geq K \] for any testing map \( \Phi \),

then

\[ \text{Re}(p(x,D)u,u) \geq c \alpha |u|^2 \quad u \in L^2(\mathbb{R}^n). \]

From Theorem 1, one can easily read off the following special case of the theorem of Hörmander (4) on commutators of vector fields:

**COROLLARY.** Let \( X_1, \ldots, X_m \) be vector fields on \( \mathbb{R}^n \) whose Lie brackets up to order \( k \) generate the Lie algebra at each point. Then

\[ \sum_{|\alpha| = 1}^m \|X_\alpha u\|^2 + C_\alpha u\|^2 \geq c |u|^2 - r+1/(k+1) > 0 \]

which \( u \in C^k \) is supported in the unit ball in \( \mathbb{R}^n \).

In fact, inequality 4 holds for \( c = 0 \), as was proved by Rothschild and Stein (5) (together with estimates in norms other than \( L^2 \)); we shall also derive that result from a refinement of Theorem 1 to be given below.

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(iii) $(\text{diam}_x Q_1)(\text{diam}_t Q_1) \leq K^{1/2}/A$.

Here $A$ is a large constant, and $Q_1^*$ is the dilate of $Q_1$ by a large constant factor. In view of the $S^m_{\Psi^m}$ calculus (see ref. 6), inequality 3 holds for $p(x, \xi)$ if and only if localized estimates hold for $p|Q_1$. Thus inequality 3 is evidently false if there is any $Q_1$ satisfying iii. Otherwise, because the localized estimate is obviously true for $Q_1$ satisfying i, the only delicate case is ii.

However, a suitable canonical transformation carries the symbol $p|Q_1$ to a symbol of the form 7, so that Theorem 2 reduces the problem to an eigenvalue computation in fewer variables.

The estimate 4 with $\varepsilon = 0$ can be obtained from our algorithm, which in fact shows that if $p = \sum p_l^2$ and

$$||p_1, p_1, \ldots, p_l, p_{l+1}, \ldots|| \geq K^{(l+1)/2}$$

for some $l$, then inequality 3 holds, the reason being essentially that the derived symbol of $p$ arising from a cube $Q_1$ of type ii is again a sum of squares satisfying hypotheses analogous to inequality 9.

Theorem 2 in turn can be deduced from the following result on the spectral decomposition of pseudo-differential operators, which may be of intrinsic interest:

**Theorem 3.** Given $p(x, \xi) \geq 0$ and a constant $K$, let

$$p_K(u) = (\min[K, p^*(x, D)]u, u),$$

in which $p^*(x, D)$ is defined by the Weyl calculus as in Hörmander (2), and the minimum is taken in the sense of spectral theory. Then if $p, q$ are nonnegative symbols satisfying inequality 1, we have

$$(p + q)_K(u) \leq C_0[p_K(u) + q_K(u) + M\|u\|^2].$$

The proofs of the results announced here will appear in a forthcoming article.

It would be interesting to know whether the lower bound for the least eigenvalue of $p(x, D)$ given by Theorem 1 is sharp.

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