Some Points of Analysis and Their History

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CHAPTER 6

The Mathematics of
Wiener’s Tauberian Theorem

Introduction

Wiener’s long paper (1932) is one of the true classics of pre-war mathematics. The main result, one of the first results in the paper, says that the closed linear hull of translates of a function

\[ f(x) \in L^1 = L^1(R) \]

is the whole space \( L^1 \) if and only if its Fourier transform

\[ F(x) = \mathcal{F}f(x) = \int_R e^{-ixt} f(t) dt \]

never vanishes. This basic result has been called Wiener’s Tauberian theorem, an unnecessarily complicated name whose origin will be explained below.

Note that the closed linear hull in question contains all convolutions

\[ f * g(x) = \int f(x-y)g(y)dy. \]

In fact this holds when \( g \) is a continuous function with compact support and we have the inequality

\[ \|f * g\| \leq \|f\| \|g\| \]

where

\[ \|f\| = \int_R |f(x)|dx. \]

About ten years after Wiener’s paper, Gelfand’s papers on normed rings or, with a later term, Banach algebras, started the field of Abstract Harmonic Analysis, which was intensely studied for almost twenty years, especially in the United States. One of the driving forces of this theory was the desire to extend Wiener’s theorem to the new abstract landscape. The book (1962) by Walter Rudin summarizes most of this research.

The name Tauberian theorem has a long and winding history. In such a theorem one knows the asymptotic behavior, say at \( x \to +\infty \), of some linear transform

\[ F(x) = \int K(x,y)f(y)dy \]

of a function \( f(x) \) and wants to deduce the asymptotic behavior at infinity of \( f \) itself. Wiener’s theorem has the following Tauberian aspect: if \( f \in L^1 \), its Fourier transform never vanishes and

\[ \lim_{x \to +\infty} \int f(x-y)g(y)dy = A \int f(x)dx \]
for some \( g \in L^\infty \), then the same holds for all \( f \in L^1 \). In fact, the limit relation above holds by an integration if \( f \) is replaced by any convolution

\[
f \ast h(x) = \int f(x - z)h(z)dz,
\]

where \( h \) is continuous with compact support and then, by taking the limit when \( f \) is replaced by any function in \( L^1 \). Then, if \( g \) is suitably regular at \( \infty \), we may conclude that \( g(t) \) tends to \( A \) for large \( t \).

The point of Wiener's paper (1932) is to deduce a great number of Tauberian theorems from Wiener's own theorem. One of them implies the prime number theorem, which, however, is proved more simply by Ikehara's theorem: if \( d\mu(x) \geq 0 \) is a measure, if the integral

\[
F(s) = \int_0^\infty e^{-sx}d\mu(x)
\]

converges for Re \( s > 1 \), if \( F(1 + it) \neq 0 \) when \( t \) is real and not zero and if

\[
F(s) \sim A/(s - 1)
\]

at \( s = 1 \), then \( \mu(t)e^{-t} \) tends to \( A \) as \( t \to \infty \).

Our interest will be Wiener's theorem in itself, properly named Wiener's density theorem. In this lecture, I shall reproduce what is essentially Wiener's original proof and compare it to the later, similar results in the theory of Banach algebras and in abstract harmonic analysis.

**Generalities about the Fourier transform**

The Fourier transform \( \mathcal{F} \) on the real line has the inverse \( \mathcal{F}^{-1} \) defined by

\[
\mathcal{F}^{-1}g(x) = \frac{1}{2\pi} \int e^{xt}g(t)dt,
\]

for suitable classes of functions. If, for instance, \( f \in L^2 \) is smooth and small at infinity, it is well known that the same holds for \( F = \mathcal{F}f \) and we have \( f = \mathcal{F}^{-1}F \).

For this class of functions the following inequalities are proved by integrations by parts:

\[
|f(t)| \leq \int |F(x)|dx, \quad |\dot{f}(t)| \leq \int |dF(x)|, \quad f = \mathcal{F}^{-1}F;
\]

in particular,

\[
(1 + t^2)|f(t)| \leq \int (|F(x)|dx + |dF'(x)|).
\]

By a passage to the limit this subsists for certain integrable functions \( f \) and all functions \( F \) in the class \( B_0 \) of continuous functions with compact supports and second derivatives of bounded variation. Hence \( B_0 \) is contained in \( TL^1 \).

It is clear that the class \( B_0 \) contains all piecewise linear functions with compact supports, in particular also the piecewise linear function \( P(x) \) which equals 1 when \( |x| \leq 1 \) and vanishes when \( |x| \geq 2 \). Note that the sum

\[
\sum_{-N}^N P(x + 2k)
\]
equals 1 when \( |x| \leq 2N \). The function \( P(x) \) and its translates will play a key role in the sequel. Since each of them equals 1 on some interval, they will be called local units.

The perfect situation occurs for \( L^2(\mathbb{R}) \). By Plancherel's theorem \( \mathcal{F} \) is unitary map on this space. In the theory of distributions it is proved that \( \mathcal{F} \) is a bijection on the space of tempered distributions.

For the space \( L^1 = L^1(\mathbb{R}) \) that Wiener considered, the situation is less symmetrical. All that can be said of the space \( TL^1 \) is that its elements are uniformly continuous and vanish at infinity.

The space \( L^1 \) is a Banach space with the norm
\[
\|f\| = \int |f(x)| \, dx.
\]
Its dual is the space \( L^\infty \) of essentially bounded functions. The crucial property of \( L^1 \) is that it is an algebra under convolutions
\[
f \ast g(x) = \int f(x-y)g(y) \, dy, \quad \|f \ast g\| \leq \|f\| \|g\|,
\]
and we have
\[
\mathcal{F}(f \ast g) = \mathcal{F}f \mathcal{F}g, \quad \mathcal{F}(e^{i\alpha x}f(x)) = \mathcal{F}f(x - \alpha).
\]
All this follows from Fubini's theorem. Hence the transform \( A = \mathcal{F}L^1 \) is a ring under pointwise multiplication. As before, its elements will be denoted by capital letters, \( F = \mathcal{F}f \) and so on. If we transport the norm on \( L^1 \) to \( A = \mathcal{F}L^1 \) so that \( \|F\| = \|f\| \), we get
\[
\|FG\| = \|F\| \|G\|.
\]
Multiplication by an exponential, \( f(t) \rightarrow e^{ita}f(t) \), is a linear isometry of \( L^1 \) and translation \( (T_aF)(x) = F(x + a) \) is a linear isometry of \( A \).

The subset \( A_0 \) of \( A \) whose elements have compact supports, in particular the space \( B_0 \) above, will play an important part in the sequel.

**Lemma 1.** \( A_0 \) is dense in \( A \).

**Proof.** Let \( 0 \neq F \in A_0 \). Then all the translates of \( F \) are in \( A_0 \) and also all products \( e^{ita}F(t) \) for real \( a \). Hence, if \( g \in L^\infty \) is orthogonal to \( T^{-1}A_0 \), we have
\[
\int e^{ita}f(t-a)g(t) \, dt = 0
\]
for all real \( b \) and \( a \). It follows that \( f(t-a)g(t) = 0 \) for almost all \( f \) and a countable dense set of values of \( a \). Since \( f \neq 0 \), this shows that \( g(t) = 0 \) almost everywhere. Hence \( A_0 \) is dense in \( A \).

**Proof of Wiener's theorem**

In view of the preceding lemma, Wiener's theorem now takes the following form in which it will be proved:

**Theorem W.** If \( F \in A \) and \( F(x) \neq 0 \) for all \( x \), then \( AF \) is dense in \( A \).

Note that if \( F \) has a zero \( x_0 \), then all elements of \( AF \) vanish at \( x_0 \) and hence the closure of \( AF \) is not equal to \( A \).

The proof depends on the following
LEMMA. Suppose that \( F \in A \) and that \( G \in B_0 \). Then
\[ \|(F(x) - F(0))G(x/\varepsilon)\| \]
tends to zero as \( \varepsilon > 0 \) tends to zero.

PROOF. The function \( G(x/\varepsilon) \) is the Fourier transform of \( \varepsilon g(\varepsilon t) \). Hence \( (F(x) - F(0))G(x/\varepsilon) \) is the Fourier transform of
\[ \int f(s)\varepsilon g(\varepsilon(t-s))ds - \varepsilon g(\varepsilon t) \int f(s)ds. \]
After a change of variables \( t \to t\varepsilon \), its norm is
\[ \int dt \int |f(s)(g(\varepsilon(t-s)) - g(\varepsilon t))|ds. \]
By dominated convergence, the double integral tends to zero as \( \varepsilon \to 0 \).

PROOF OF THE THEOREM. Let \( P(x) \) be the local unit defined above, i.e. \( P(x) \) is a piecewise linear function which vanishes when \( |x| > 2 \) and equals 1 when \( |x| < 1 \). Consider the quotient
\[ \frac{P(x/\varepsilon)}{F(x)} \]
Note that, by assumption, \( F(x) \neq 0 \) for all \( x \). When \( P(x/\varepsilon) \neq 0 \), then \( P(x/2\varepsilon) \) equals one and hence we may rewrite the quotient as
\[ \frac{P(x/\varepsilon)}{F(0) + P(x/2\varepsilon)(F(x) - F(0))}. \]
By the lemma, the norm of
\[ G_\varepsilon(x) = P(x/2\varepsilon)(F(x) - F(0)) \]
tends to zero with \( \varepsilon \). Hence, if \( \varepsilon \) is small enough, we can express our last quotient as a geometric series. The result is that
\[ P(x/\varepsilon) = \frac{F(x)}{F(0)} \sum_{0}^{\infty} (-1)^{k}(G_\varepsilon(x)/F(0))^k \]
where the series converges so that the right side belongs to \( A \). In other words, \( P(x/\varepsilon) \in F(x)A \) when \( \varepsilon > 0 \) is sufficiently small. Since \( |F(x)| \) has a positive lower bound on every compact interval, all translates \( P(x+\varepsilon) \) of \( P(x/\varepsilon) \) belong to \( FA \) when \( b \) is bounded and \( \varepsilon \) is sufficiently small.

Now \( P(x/\varepsilon) \) equals 1 in the interval \( |x| \leq \varepsilon \) and a sum of suitable translates will be 1 in any interval. Hence to every interval \( |x| \leq N \) there is a function in \( FA \) which equals 1 on this interval. But this obviously means that any element of \( A_0 \) belongs to \( FA \). This finishes the proof of Wiener's Tauberian theorem.

Wiener's theorem for Fourier series

Let \( L^1(Z) \) be the space of sequences \( f = (f(n)) \) labelled by the integers \( n \) and such that
\[ \|f\| = \sum |f(n)| < \infty. \]
The Fourier transform \( F = \mathcal{F}f \) of \( f \) is then a continuous function
\[
F(x) = \sum f(n)e^{-in\pi}
\]
in the interval \(|x| < \pi\) such that \( F(-\pi) = F(\pi) \). We can also consider \( F(x) \) to be a continuous function on the unit circle. Hence the previous real line is replaced by a compact set. Let \( A \) be the space of these functions with the norm
\[
\|F\| = \|f\|.
\]
In other words, \( A \) is the space of \( 2\pi \)-periodic functions with convergent Fourier series. In contrast to the previous case, the ring \( A \) now contains a unit 1.

We can now repeat the previous argument with the functions \( P(x/\varepsilon) \) and prove Wiener's result that \( 1/F(x) \) has a convergent Fourier series when \( F(x) \neq 0 \) everywhere.

**Remark.** Actually, Wiener used a different argument based on the fact that if
\[
|f(0)| > \sum_{n \neq 0} |f(n)|,
\]
then \( F(x) \neq 0 \) everywhere and \( 1/F(x) \) can be written explicitly as a function in \( A \). He then showed that if \( F(0) \neq 0 \), the constant term dominates in the same way the Fourier series of
\[
F_{\varepsilon}(x) = P(x/\varepsilon) + P(0)(1 - P(x/\varepsilon))
\]
when \( \varepsilon > 0 \) is sufficiently small. Hence \( 1/F_{\varepsilon}(x) \) belongs to \( A \). Since \( F_{\varepsilon}(x) = 1 \) close to 0, Wiener could then complete the proof by the same arguments as above.

**Remark.** Wiener's proof of the density theorem for Fourier integrals uses the results for Fourier series, now for functions which are periodic in a large interval \( I \) and vanish close to the endpoints. In this case the Fourier series for such functions approximate their Fourier integrals when the interval tends to the entire real line.

**Normed rings**

The space \( L^1(Z) \) is a Banach space with norm \( \|f\| \) and also a commutative ring under convolution \( f \ast g \) such that \( \|f \ast g\| \leq \|f\| \|g\| \). Moreover, it has a unit \( e \) given by \( e(n) = \delta(n) \). We have a Banach space which is also a commutative ring with a unit.

In his influential articles (1941), I. Gelfand investigated such objects which he called normed rings, i.e. rings \( R = (e, a, b, c, \ldots) \), which are complex Banach spaces with a norm \( |a| \), a commutative multiplication \( ab = ba \) such that \( |ab| \leq |a||b| \), and a unit \( e \) of norm one. The geometric series
\[
\frac{e}{a-b} = a^{-1} \sum_{k=0}^{\infty} b^k a^{-k},
\]
where \( a \) is invertible and \( b \) is small, shows that the set \( E \) of invertible elements is open. If \( R \) is a field, i.e. all elements except zero are invertible and an element \( a \) is not a complex multiple of \( e \), then
\[
(e + za)^{-1}
\]
is an entire analytic function with values in $R$ and hence a constant which must be zero. This contradiction shows that a normed field consists of all complex multiples of the unit.

In the general case, the main trick is now to consider ideals of $R$, by definition not equal to $R$. Any such ideal does not intersect the ball $E : |e - a| < 1$ and hence is contained in the closed set $R - E$. It follows that the closure of an ideal is an ideal and so is the union of any ascending chain of ideals. Hence we have the following observation which relies heavily on the existence of a unit, namely

*Every ideal is contained in a maximal ideal.*

Here an ideal $I$ is said to be maximal when there is no ideal strictly between $I$ and $R$. If $I$ is a maximal ideal, the quotient $x = R/I$ is a field and conversely. Hence the quotient map $a \rightarrow a(x)$ can be identified with a homomorphism $a \rightarrow a(x)$ into the complex numbers such that

$$(ab)(x) = a(x)b(x), \quad |a(x)| \leq |a|,$$

where the ideal $I$ consists of all $a$ such that $a(x) = 0$.

We can now define a maximal ideal space $X$ as a set of points $x$, each one corresponding to a maximal ideal. The first main theorem of Gelfand's theory is now

**Theorem.** An element $a \in R$ is invertible if and only if $a(x) \neq 0$ for all $x$ in the maximal ideal space.

In fact, $Ra$ is then not contained in any maximal ideal and must be $R$ itself.

**Application to absolutely convergent Fourier series**

When $R = L^1(Z)$, what is the maximal ideal space $X$? To answer this question, let us note that $L^1(Z)$ has a generator $g$ of norm 1, with an inverse of norm 1 given by $g(n) = \delta(n - 1)$ with the inverse $g^{-1}(n) = \delta(n + 1)$. The elements of $R$ have the form

$$a = \sum c_n g^n, \quad |a| = \sum |c_n|.$$

If $x = R/I$ and $I$ is a maximal ideal, it follows that $|g(x)| \leq 1$ and $|g^{-1}| \leq 1$. Hence $|g(x)| = 1$ and

$$a(x) = \sum c_n g(x)^n$$

is indeed a homomorphism of $R$ into the complex numbers. Writing $a(x) = e^{it}$ we have recovered the absolutely convergent Fourier series and Wiener's theorem.

The convolution ring of absolutely convergent measures also has a unit, and here Gelfand's methods give a very simple proof of a theorem by Wiener and Pitt, which says that a measure is invertible if the total measure of its singular part is less than the lower bound of the absolute values of its Fourier transform. For $L^1(R)$, which is a ring without a unit, this simple reasoning with maximal ideals does not work and additional arguments are necessary. We shall not go into this nor shall we comment on the rest of Gelfand's papers listed in the bibliography. Instead we shall return to $L^1(R)$ and Wiener's theorem but now with a new angle and with methods which apply to $R^n$ and other groups.
Fourier analysis on locally compact Abelian groups

The possibility of extending Fourier analysis from the real line to locally compact abelian groups was pointed out by André Weil in a very influential book (1938). If $G = \{x, y, \ldots\}$ with addition $x + y$ and inverse $-x$ is such a group, it has a measure $\mu \geq 0$ which is finite on compact subsets $C$ and invariant in the sense that $\mu(C + y) = \mu(C)$. We can then form the space $L^1(G)$ of integrable functions with finite norm

$$\|f\| = \int_G |f(x)|d\mu(x).$$

For such functions there is a commutative and associative convolution $f * g$. The dual group $\Gamma$ is the space of characters, continuous homomorphisms $x \mapsto (x, \xi)$ of $G$ into the unit circle $U$. Examples: if $G = \mathbb{R}^n$, $\Gamma = \mathbb{R}^n$, if $G = \mathbb{Z}^n$, $\Gamma = \mathbb{U}^n$, if $G$ is discrete, $\Gamma$ is compact.

The Fourier transform

$$\mathcal{F}f = \int f(x)e^{ix\xi}d\mu(x)$$

then maps $L^1(G)$ into the subspace of $L^\infty(\Gamma)$ whose elements vanish at infinity in $\Gamma$. The ordinary Fourier integral and Fourier series are special cases of this theory. In the forties, many efforts were made to extend classical harmonic analysis to the general framework of locally compact abelian groups.

Extension of Wiener's theory

We shall extend Wiener's theory to locally compact abelian groups. To simplify, we reason with $G = \mathbb{R}^n$ and $\Gamma = \mathbb{R}^n$ in such a way that the results extend to the general case.

To begin with, we shall construct local units.

Local units and their decomposition

In the sequel we shall write the Fourier transform of $f \in L^1 = L^1(G)$ as

$$\hat{f}(\xi) = \int e^{ix\xi}f(x)dx, \quad x\xi = \sum x_k\xi_k,$$

and differentiate between $G = \{(x, y, \ldots)\}$ and $\Gamma = \{(\xi, \eta, \ldots)\}$, connected by the duality $x\xi$. Let $A$ be the set of Fourier transforms of functions in $L^1$.

A local unit is now a function in $L^1$ whose Fourier transform has compact support and equals 1 on some non-empty open set $U \subset \mathbb{R}^n$.

**Lemma 1.** Given a bounded open non-empty subset $V \subset \mathbb{R}^n$, there is a local unit $F \in L^1$ such that $\hat{F}$ is supported in $V$ and equals 1 on a neighborhood $U$ of the origin. Moreover,

$$\|F\| \leq 2c, \quad c = (2\pi)^n,$$

and

$$\| (T_x - 1)F \| \leq 4c \sup \{ |e^{ix\xi} - 1| \}$$

where $T_x$ is translation by $x$, $(T_x F)(y) = F(x + y)$. 

Note that (2) means that \( F(x) \) is close to 1 on any compact set when \( V \) is small.

**Proof.** There are open sets \( W \) such that \( W \subset V \) and hence also an open neighborhood \( U \) of the origin such that \( W \pm U \subset V \). Let \( |V| \) be the volume of \( V \) and similarly for \( |W| \) and choose \( W \) such that \( 4|W| > |V| \). Let \( \hat{f}, \hat{g} \) be the characteristic functions of \( V \) and \( W \). We shall see that
\[
F(x) = \frac{f(x)g(x)}{|W|}
\]
has the required properties. In fact \( \|f\|_2 = \sqrt{|V|} \) and \( \|g\|_2 = \sqrt{|W|} \) so that by Parseval's formula,
\[
\|F\| \leq \|f\|_2 \|g\|_2 / |W| \leq c \sqrt{|V| / |W|} \leq 2c
\]
and, since
\[
\hat{F}(\xi) = \int \hat{f}(\xi - \eta)\hat{g}(\eta)d\eta,
\]
\( \hat{F} \) is supported in \( V \) and equals 1 on \( U \). Also
\[
|W| \| (T_x - 1)F \| \leq \| (T_x - 1)f \|_2 \|g\|_2 + \|f\|_2 \| (T_x - 1)g \|_2.
\]
Since the Fourier transform of \( (T_x - 1)f \) is \( (e^{ix} - 1)\hat{f}(\xi) \), insertions of \( \|f\|_2 \) and \( \|g\|_2 \) prove (2).

The proof of the next lemma is obvious.

**Lemma 2.** If \( U \) is an open set covered by a union of finitely many open sets \( U_1, \ldots, U_m \), and \( F_1, \ldots, F_m \) are local units such that \( F_k \) equals 1 on \( U_k \), then
\[
1 - (1 - F_1) \cdots (1 - F_m)
\]
is a local unit equal to 1 on \( U \).

**Modules for \( L^1 \) and Beurling's theorem**

The dual of \( L^1 = L^1(G) \) is \( L^\infty \) and if \( g \in L^\infty \), \( f \in L^1 \), the convolution
\[
f \ast g \in L^\infty
\]
is well defined. In other words, \( L^\infty \) is an \( L^1 \)-module.

Let us now define spectra and null sets for functions in \( L^1 \) and \( L^\infty \).

**Definition.** A point \( \xi \in \Gamma \) belongs to the spectrum \( \text{sp}(g) \) of a function \( g \in L^1(G) \) or \( L^\infty(G) \) if, given any neighborhood \( N \) of \( \xi \), there is \( f \in L^1 \) such that \( \hat{f} \) is supported in \( N \) and \( f \ast g \neq 0 \). If, on the contrary, \( f \ast g = 0 \) under the same conditions, \( \xi \) belongs to the null set \( \text{N}(g) \) of \( g \).

**Remark.** It follows from the definition that \( \text{sp}(g) \) and \( \text{N}(g) \) are closed but that their interiors have an empty intersection. When \( g \in L^1 \), the Fourier transform of \( f \ast g \) is \( \hat{f}\hat{g} \) and hence \( \text{sp}(g) \) is simply the support of \( \hat{g} \). When \( g \in L^\infty \), \( \text{sp}(g) \) is also the support of the Fourier transform of \( g \), regarded as a distribution. But since our arguments are meant to apply to locally compact abelian groups, we cannot use the notion of distribution. Still we may imagine that \( \text{sp}(g) \) is the support of its virtual Fourier transform.
Multiplication by an exponential translates a spectrum. In fact, the Fourier transform of $e^{ix\eta}g(x)$ is $\hat{g}(\xi - \eta)$ when $g \in L^1$ and a small computation gives the same result when $g \in L^\infty$. Similarly: a translation in $G$ corresponds to multiplication by an exponential in $\Gamma$.

**Beurling's theorem**

We can now prove a slight extension of a theorem by Beurling (1945) which implies Wiener's theorem.

**Theorem B.** Suppose that $g \in L^\infty$ and that $\xi \in \text{sp}(g)$. Suppose further that $F_n$ is a bounded sequence of local units whose Fourier transforms equal 1 close to $\xi$, and such that their spectra tend to $\xi$. Then all the limits of the functions $F_n * g$ under locally uniform convergence are multiples of $e^{ix\xi}$.

**Remark.** This implies Theorem W, for if the Fourier transform of $f \in L^1$ never vanishes and $f * g = 0$ for some $0 \neq g \in L^\infty$, then $\text{sp}(g)$ must contain a point $\xi$ so that $\hat{f}(\xi) = 0$ which is a contradiction. Hence $f * g = 0$ implies that $g = 0$ so that $L^1 * f$ must be dense in $L^1$. But Beurling’s theorem is a bit stronger.

**Remark.** In the original paper, Beurling distanced himself from Wiener by phrasing his theorem as a property of bounded and uniformly continuous functions $f(x)$ on the real line, namely the following: if $f$ does not vanish identically, then there are an exponential $e^{ix\xi}$ and linear combinations of translates,

$$g(x) = \sum c_k f(x - x_k),$$

which tend to the exponential uniformly on compact sets and so that $\sup |g(x)|$ tends to 1. In the proof below and under the only condition that $f(x)$ is essentially bounded and not zero, there are convolutions

$$\int f(x - y)g(y)dy, \quad g \in L^1,$$

which tend to the exponential in the manner indicated. When $f(x)$ is also uniformly continuous, such integrals can be approximated in the same way by linear combinations above.

**Proof.** By a translation of $\text{sp}(f)$, we may assume that $\xi = 0$. Let $G_n$ be local units constructed in Lemma 1 such that their spectra tend to 0 and $G_n * F_n = F_n$. Then no $g_n = G_n * g$ vanishes, the sequence $\|g_n\|$ is bounded, $\text{sp}(g_n)$ tends to zero and, by the lemma,

$$|g_n(x) - g_n(0)| = |F_n * g_n(x) - F_n g_n(0)| \leq \|(T_x - 1)F_n\| \|g\|_\infty$$

tends to zero locally uniformly as $n \to \infty$. Hence every subsequence of $(g_n)$ has a subsequence $(g_{n'})$ for which $g_{n'}(0)$ converges and hence $g_{n'}(x)$ is a bounded sequence of functions which tends to a constant $c$, uniformly on compact subsets.

**Remark.** If we normalize so that $\|g_n\|_\infty = 1$ and translate $g_n$ so that, for instance,

$$1 \geq g_n(0) \geq 1/n,$$

this sequence has norm 1 and tends to 1 locally uniformly. Note that if $\text{sp}(g)$ consists of 0 alone, all $g_n(x)$ are the same and hence equal to some constant $\neq 0$. 
Spectra and null sets of ideals. Two ideals with the same null set

The null set $N(I)$ of an ideal $I$ is defined as the intersection of all $N(f)$ for $f \in I$; the spectrum $\text{sp}(I)$ is the union of all $\text{sp}(f)$. Both these subsets of $\Gamma$ are closed and the same for an ideal and its closure in $L^1(G)$.

After these definitions, the following problem seems natural: are there two different closed ideals with the same null set? In 1948 Laurent Schwartz proved very simply that the answer is no when the null set is the unit sphere $E$ in $\Gamma = R^3$. In fact, the inverse Fourier transform of the unit rotation invariant measure $d\mu(\xi)$ on $E$ turns out to be $\sin|x|/|x|$. Hence, if $f$ and $\hat{f}$ are smooth functions, then

$$\int f(x) \frac{x_1}{|x|} \sin |x| dx = \text{const} \int \frac{\partial \hat{f}(\xi)}{\partial \xi_1} d\mu(\xi).$$

The left side is a continuous function of $f \in L^1$ which vanishes on the closure of the set $I$ of all smooth functions $f$ such that

$$\hat{f}(\xi) = 0, \quad \frac{\partial f(\xi)}{\partial \xi_1} = 0$$

on $E$ but does not vanish on the set $J$ of smooth functions which satisfy just the first equality. Moreover, both $I$ and $J$ are translation invariant. Hence $N(I) = N(J) = E$ but $I$ and $J$ are not the same ideal.

When does an element belong to an ideal?

After the counterexample above, it seems interesting to know general criteria involving spectra which guarantee that a given function belongs to a given ideal. We shall prove one such theorem which follows from Beurling's theorem.

**Theorem.** If $I \subset L^1$ is a closed ideal, $f \in L^1$ and $N(I) \cap \text{sp}(f)$ is countable, then $f \in I$.

**Remark.** Since $N(I)$ and $\text{sp}(f)$ are closed, their intersection is a closed, countable set.

**Proof.** By duality it suffices to prove that

$$I * g = 0 \implies f * g = 0$$

for every $0 \neq g \in L^\infty$. By the first equality, the spectrum of $g$ has an empty intersection with the interior of the spectrum of $I$. In other words, $sp(g) \subset N(I)$.

By assumption, $N(I) \cap \text{sp}(f)$ must have isolated points $\xi$. Hence if $H$ is a local unit with support close enough to $\xi$, the function $H * f * g$ is actually independent of $H$, i.e. $H_1 * f * g = H_2 * f * g$ when $H_1, H_2$ are as $H$. By Theorem B, there is a sequence $H_n$ of local units whose spectra tend to $\xi$ such that the limits of $H_n * g$ are multiples of $e^{i\alpha \xi}$ and hence the limit of $H_n * f * g$ must be a multiple of $\hat{f}(\xi) = 0$. Hence $H * f * g = 0$ when $H$ is a local unit whose spectrum is close enough to $\xi$.

Since every subset of $N(I) \cap \text{sp}(g)$ must have isolated points, it follows that $H * f * g$ vanishes for all local units with spectra close to $N(I) \cap \text{sp}(g)$. Hence the same holds for all local units and hence $f * g = 0$. 


Permitted sets

When \( C \subset \Gamma \) is closed, let \( I(C) \) be the ideal of functions \( f \in L^1(G) \) such that \( \hat{f} \) vanishes on \( C \). Let us say that \( C \) is permitted if \( I(C) \) is the only ideal whose null set is \( C \).

The theorem just proved shows that a finite collection of points in \( G \) or a finite collection of closed intervals in \( R \) is permitted but not much more. But many more sets are permitted, at least in \( \Gamma = R^n \), for instance radial sets with the property that they contain a point \( \xi \) such that

\[
\bigcap_{n \geq 1} a(C - \xi) \supset C - \xi
\]

for some \( \xi \). In the proof we can take \( \xi = 0 \). Suppose that \( \hat{f} \) vanishes on \( C \).

Then \( \hat{f}(\xi/a) \) vanishes in a neighborhood of \( C \) and is the Fourier transform of \( f_a(x) = a^n f(ax) \) which belongs to \( I(C) \). Since \( ||f_a - f|| \) tends to zero as \( a \) decreases to 1, \( f \) itself belongs to \( I(C) \).

We have now come to the end of our journey from Wiener’s theorem to abstract harmonic analysis, once a subject of intense interest. Rudin’s book (1962) contains among many other things a proof of Malliavin’s negative result: there are closed sets in every non-discrete abelian group which are not permitted.

Bibliography

BEURLING A.

GELFAND I.

MALLIavin P.

RUDIN W.

WEIL A.

WIENER N.